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ON THE SINGULAR CARDINALS PROBLEM I

BY

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ABSTRACT

We show how to get a model of set theory in which $\aleph_\omega$ is a strong limit cardinal which violates the generalized continuum hypothesis. Generalizations to other cardinals are also given.

§0. Introduction

The singular cardinals problem is a direct descendent of the generalized continuum problem, namely: *Given the cardinality of the set* A, *what is the cardinality of the power set of* A ($P(A)$ is the power set of A, $|P(A)| = 2^{|A|}$ where $|A|$ is the cardinality of A). The Generalized Continuum Hypothesis (G.C.H.) is the statement which asserts that $2^{|A|}$ is always $|A|^+$ where $|A|^+$ is the cardinal which is the immediate successor of $|A|$.

It is well known that the accepted system of axioms for set theory does not settle this problem ([2]). Thus the power set of the set of natural numbers, for instance, can have different cardinalities according to the particular model which we happen to consider. Similar results can be derived about larger sets.

Naturally one wonders whether, given such freedom for deciding what is the cardinality of the power set, there are nontrivial theorems about the relation between $|A|$ and $2^{|A|}$ or there are no deep facts about this relation which follow from the axioms. By classical methods the following theorems were known:

(I) *If* $|A| \leqq |B|$ *then* $2^{|A|} \leqq 2^{|B|}$.

(II) *The cofinality of* $2^{|A|}$ *is greater than* $|A|$.

For definitions of "cofinality" and "singular cardinal" see §1. (II) is Konig's Inequality. Cantor's Theorem, i.e. $2^{|A|} > |A|$, follows from (II).

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(Bukovsky [1]) *If* α *is a singular cardinal and for some* λ *and*

(III) $\gamma < \alpha$, $\gamma < \beta < \alpha$ *implies that* $2^\beta = \lambda$ *then* $2^\alpha = \lambda$.

See also Jech [7] for more facts about powers of singular cardinals.

Easton [4] showed that in some sense (I) and (II) are all the facts we can prove in the Zermelo–Fraenkel set theory (ZFC) about 2^α provided we restrict our attention to *regular cardinals.* Roughly his result can be stated as follows. Every function F defined on the regular cardinals such that F is non-decreasing and for every α the cofinality of $F(\alpha)$ is greater than α (i.e. F satisfies (I) and (II)) is a possible candidate for being the function $\alpha \to 2^\alpha$, i.e., there is a model of set theory in which $2^\alpha = F(\alpha)$ holds for all regular cardinals.

What happens in the models constructed by Easton to powers of singular cardinals? If α is a regular cardinal then in the Easton model 2^α is the minimal cardinal for which (I), (II) are not violated. Thus for instance, if $2^{\aleph_n} < \aleph_\omega$ for every $n < \omega$ then in the Easton model $2^{\aleph_\omega} = \aleph_{\omega+1}$. If $2^{\aleph_n} = \aleph_{\omega+n+1}$ then $2^{\aleph_\omega} = \aleph_{\omega+\omega+1}$. (Note that $2^{\aleph_\omega} = \aleph_{\omega+\omega}$ is ruled out by (II).) Exactly the same behaviour of powers of singular cardinals occurs in another case. A theorem of Solovay ([24]) asserts that if α is a singular cardinal which is above a strongly compact cardinal then 2^α is again the minimal cardinal β such that "$2^\alpha = \beta$" is consistent with (I), (II). For the definition of "super compact cardinal" which is a stronger concept than "strongly compact", see §1.

These two cases could have created the suspicion that this behaviour is a theorem of ZFC and hence there is no singular cardinal problem, but Solovay's Theorem was discovered, it was after the combined construction of Prikry [16] and Silver [22] indicated that is not a theorem of ZFC. Silver described a method for getting a model in which some measurable cardinal violates the G.C.H. Prikry gives a forcing notion which changes the cofinality of a measurable cardinal to ω without collapsing any cardinals or changing the function $\alpha \to 2^\alpha$. Hence we get a model which a singular cardinal of cofinality ω satisfies $2^\alpha > \alpha^+$, while α is a strong limit cardinal, i.e., for $\beta < \alpha$ $2^\beta < \alpha$. Note that for strong limit α, the minimum value for 2^α which is consistent with (I), (II) and (III) is α^+.

The cardinal obtained by the Prikry–Silver construction is rather large (though singular) and is greater than each one of $\aleph_\omega$, $\aleph_{\omega_1}$, the first α such that $\alpha = \aleph_\alpha$, etc. The question whether similar results can be obtained for more down to earth singular cardinals (like $\aleph_\omega$, $\aleph_{\omega+\omega}$, $\aleph_{\omega_1}$, etc.) poses itself, and this paper gives some information about it. Probably a more interesting and important problem is raised by the well known fact (due to Scott [19]) that if a measurable cardinal violates the Generalized Continuum Hypothesis, then there are many smaller cardinals which violate G.C.H. Hence when we apply the Prikry–Silver construction, the singular cardinal we get has many cardinals below it which violates G.C.H. The question is whether this can be avoided, namely, can a singular cardinal be the first cardinal which violates G.C.H.? Compare this problem with the similar problem for regular cardinals, where by Easton's results "any" regular cardinal can be the first which violates G.C.H. ("any" here means definable in some way which is absolute for a certain class of Cohen extensions. Of course the "second regular cardinal which violates G.C.H." cannot be the first).

Silver's surprising result ([22]) shows that the answer to our problem is "No" for singular cardinals of cofinality greater than ω. In a forthcoming paper ([13]) we shall address ourselves to this problem for cardinals of cofinality ω and show that the answer can be "Yes" even for the least singular cardinal of cofinality ω, i.e., $\aleph_\omega$. In both papers we assume the consistency of the existence of some very large cardinals, in this paper we assume the consistency of the existence of a supercompact cardinal and in [13] even the stronger assumption of the consistency of the existence of a huge cardinal. (This difference in the assumptions, as well as some difference in methods and the fact that here we can generalize the results to cardinals of cofinality greater than ω, whereas in [13], in view of Silver's Theorem, no such generalization is possible, are the main reasons for separating the two papers.) These strong infinity assumptions are necessary, at least to a certain extent as follows from the following important results of Jensen [3]. If the axiom "$\forall a$ (a is real $\rightarrow \exists a^{\#}$)" (which is related to strong cardinal assumptions and is inconsistent if, for instance, the existence of ineffable cardinals is inconsistent) fails then for every singular α, 2^α is simply the minimal value consistent with (I), (II). The assumption (a is real $\rightarrow \forall a\ \exists a^{\#}$) is still much weaker than the existence of measurable or super compact cardinals, but a recent result by Jensen and Dodd is that the gap can be narrowed and if for some singular α, 2^α is not the minimal value possible, then there exists an inner model with a measurable cardinal. Thus large cardinals, having order of magnitude of measurable cardinals, are probably necessary for getting the results of this paper.

We are ready for the statement of our results, assuming the consistency of the existence of a super compact cardinal (actually for any particular case a weaker assumption is sufficient).

THEOREM 1. (a) *Let k be a natural number. There is a model of ZFC in which* $2^{\aleph_n} < \aleph_\omega$ *for* $n < \omega$ *and* $2^{\aleph_\omega} = \aleph_{\omega+k}$.

(b) *There is a model of ZFC in which* $2^{\aleph_n} < \aleph_\omega$ *for* $n < \omega$ *and* $2^{\aleph_\omega} = \aleph_{\omega+\omega+1}$.

Some remarks about the behaviour of the function $n \to 2^{\aleph_n}$ for $n < \omega$, in the models we construct in the proof of Theorem 1. The model constructed in the proof of Theorem 1 (a) satisfies:

$$2^{\aleph_0} = \aleph_1,\ 2^{\aleph_1} = \aleph_2,\ 2^{\aleph_2} = 2^{\aleph_3} = \cdots 2^{\aleph_{2+k-1}} = \aleph_{2+k},$$

$$2^{\aleph_{2+k}} = \aleph_{2+k+1},\ 2^{\aleph_{2+k+1}} = 2^{\aleph_{2+k+2}} = \cdots = \aleph_{2+k+1+k} \quad \text{etc.}$$

We also have some freedom and by slight changes in the proof we can have for different f's a particular model which satisfies the conclusions of Theorem 1 (a) and $2^{\aleph_n} = \aleph_{f(n)}$. For instance, we can prescribe the value of $2^{\aleph_n}$ for finitely many n's, provided we conform to requirements (I), (II) and $\forall n\ (2^{\aleph_n} < \aleph_\omega)$. But we must have an infinite set of n's such that $2^{\aleph_n} = \aleph_{n+k}$. If we wanted to get for Theorem 1 (a) a model in which $2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$ then the methods of [13] are called for. Note that, trivially, above $\aleph_\omega$ we have no commitment about the value of 2^α for regular α except that (I), (II) and (III) should be preserved.

Note also that we do not know how to get models with $2^{\aleph_\omega}$ arbitrarily large, while keeping $\aleph_\omega$ to be a strong limit cardinal. If instead of $\aleph_\omega$ we had a regular cardinal α, then a bound on 2^β for $\beta < \alpha$ gives no bound on 2^α. For singular cardinals of cofinality greater than ω there is a bound as follows from the results of Galvin and Hajnal ([5]). For instance, for $\aleph_{\omega_1}$ it follows from [5] that if it is a strong limit cardinal then $2^{\aleph_{\omega_1}} < \aleph_{(2^{\omega_1})^+}$. The following theorem (of which Theorem 1 is a special case) shows how far we get with our methods for cardinals of cofinality $> \omega$. Note that there is a wide gap from the Galvin–Hajnal bound to the maximum we get. For instance, it follows from the next theorem that we can have models in which $\aleph_{\omega_1}$ is a strong limit cardinal but $2^{\aleph_{\omega_1}}$ is any successor cardinal up to $\aleph_{\omega_1+\omega_1+1}$.

In order to state the theorem in a more general setting, we need some definitions. Let β be a limit ordinal. An additive partition of β is a non-decreasing sequence of ordinals $\langle \delta_\alpha \mid \alpha < \rho \rangle$ and an ordinal μ such that $\beta = \mu + \Sigma_{\alpha<\rho}\delta_\alpha$ where $\mu < \beta$ and $\delta_\alpha < \beta$ for $\alpha < \rho$, ρ is a limit cardinal. The caliber of an additive partition of β is $\sup(\langle \delta_\alpha \mid \alpha < \rho \rangle)$. γ is good for β if γ is a successor ordinal such that γ is less or equal than the successor of the caliber of some additive partition of β. For instance, for every limit ordinal, β, every finite γ is good for it since β can be expressed in the form $n \cdot \rho$ for some ρ. If β is a regular cardinal then every successor ordinal $\leqq \beta + 1$ is good for β. For $\beta = \omega_1 + \omega$, the maximal ordinal which is good for β $\omega + 1$, etc.

THEOREM 2. *Let M be a countable model of ZFC such that*

(a) *$M \models \kappa$ is a supercompact cardinal,*

(b) $M \models \beta$ *is a limit ordinal and* $\beta < \kappa$,

(c) $M \models \gamma$ *is good for* β.

Then there exists a Cohen extension of M, N *in which* $\aleph_\beta$ *is a strong limit cardinal and* $2^{\aleph_\beta} = \aleph_{\beta+\gamma}$. *Moreover if* $|\beta| < \aleph_{|\beta|}$ *we can assume then in* N *there are no new subsets of* β.

Again like in Theorem 1 we have some freedom to determine the values in N of 2^α for $\alpha < \beta$, but this freedom is rather limited. In any case we can prescribe the value of $2^{\aleph_\alpha}$ for α's in a bounded set of successor ordinals provided we preserve (I), (II) and the fact that $\aleph_\beta$ should be a strong limit cardinal. Of course, in this case we may have to give up the clause "in N there are no new subsets of β". The reason we were interested in this requirement is because if, for instance, in M, β is ω_1, $\aleph_{\omega_1}$, $\omega_1 + \omega^2$, ω_2^2 in the sense of M, etc. We want β to be ω_1, $\aleph_{\omega_1}$, $\omega_1 + \omega^2$, ω_2^2 respectively in the sense of N.

Though Theorem 2 includes Theorem 1, for didactic reasons we shall give the full details just for the proof of Theorem 1 (a), then describe the modification of this proof needed for proving Theorem 1 (b), and finish by describing how to modify the proof of Theorem 1 (b) to get Theorem 2. Just one of the possible cases we can get by Theorem 2 is:

COROLLARY 3. *If in* M, β *is a regular cardinal,* $\beta < \aleph_\beta$, $\beta < \kappa$ *where* κ *is supercompact in* M, *then for every successor* $\gamma \leqq \beta + 1$ *there exists a Cohen extension of* M *in which* $2^{\aleph_\beta} = \aleph_{\beta+\gamma}$, β *is a strong limit cardinal and no new subsets of* β *were introduced.*

The structure of the paper is as follows: §1 introduces the notations and some preliminaries used in the rest of the paper. §2 introduces the set of the forcing conditions, using which we can get the Cohen extension which is a witness for Theorem 1 (a). The basic technical lemma about this forcing notion is proved in this section. In §3 we verify that this forcing notion does its intended job. In §§4 and 5 we indicate how to modify §§2 and 3 in order to get Theorem 1 (b) and Theorem 2 respectively.

A further corollary of Theorem 1 improves a result of Jech in [7]. The problem is the relation between the gimel function, i.e. $\alpha \to \aleph_\alpha^{\mathrm{cof}(\aleph_\alpha)}$ and the power function $\alpha \to 2^\alpha$. Bukovsky in [1] proved that the power function can be computed from the gimel function. Jech in [7] shows that the converse is false. He produces two models of set theory having the same ordinals, cardinals and power function, but for some cardinal κ having cofinality $\aleph_0$, $\kappa^{\aleph_0} = \kappa^+$ in one model and $\kappa^{\aleph_0} = \kappa^{++}$ in the second model. The cardinal κ is rather large and cannot be easily defined. Theorem 1 helps us to get Jech's counter example to more down to earth cardinals. Using the model we get by Theorem 1 (a) for $k = 2$ as a ground model for the usual Cohen extension blowing $2^{\aleph_1}$ to $\aleph_{\omega+2}$. It is well known that this Cohen extension introduces no new countable subsets of ordinals and since $2^{\aleph_0} = \aleph_1$ in the ground model no cardinals are collapsed and in the extension $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_\alpha} = \aleph_{\omega+2}$ for $0 < \alpha \leqq \omega$. In the ground model (i.e. the model supplied by Theorem 1) $\aleph_\omega^{\aleph_0} = \aleph_{\omega+2}$ ($\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} = \aleph_{\omega+2}$), therefore since no new countable sequences of ordinals were introduced by the extension $\aleph_\omega^{\aleph_0} = \aleph_{\omega+2}$ in the extension. We can also assume that G.C.H. holds for cardinals greater than $\aleph_\omega$.

On the other hand, if one applies the Easton construction to a ground model satisfying G.C.H. and makes $2^{\aleph_1} = \aleph_{\omega+2}$ without introducing new countable sequences of ordinal we have in the extension $2^{\aleph_0} = \aleph_1$, $2^{\aleph_\alpha} = \aleph_{\omega+2}$ for $\alpha \leqq \omega$, G.C.H. holds above $\aleph_\omega$, but $\aleph_\omega^{\aleph_0}$ is $\aleph_{\omega+1}$ since in the ground model $\aleph_\omega^{\aleph_0} = 2^{\aleph_\omega} = \aleph_{\omega+1}$. Hence in spite of the fact that the power function is the same in the two models the gimel function is different and the counter example is the least possible, namely $\aleph_\omega$.

This remark solves a problem of Bukovsky [1].

§1. Notations and preliminaries

We assume acquaintance with basic set theory. Jech's [6] is a very good reference to our notations and terminology. We reserve the use of lower case Greek letter, except π and χ to ordinals. κ is always a cardinal and i, j, l, n are always natural numbers. For a set A, $|A|$ is the cardinality of A. If A is a set of ordinals, then $\mathrm{otp}(A)$ is the order type of A and $\sup A$ is $\cup A$. α^+ is the least cardinal greater than α. $\alpha^{+\gamma}$ is defined by induction $\alpha^{+0} = \alpha$, $\alpha^{+(\delta+1)} = (\alpha^{+\delta})^{+1}$ and if δ is a limit ordinal $\alpha^{+\delta} = \sup\{\alpha^{+\delta} \mid \gamma < \delta\}$. α^β is the cardinality of the set of all functions from β to α. α is strong limit if $2^\beta < \alpha$ for all $\beta < \alpha$. $2^{\underline{\beta}}$ (α to the weak power β) is $\sup_{\gamma<\beta} \alpha^\gamma$. The cofinality of α ($\mathrm{cof}(\alpha)$) is the minimal ordinal β such that $\alpha = \sup A$ for some set $A \subseteq \alpha$ and $\mathrm{otp}(A) = \beta$. α is *regular* if $\mathrm{cof}(\alpha) = \alpha$, and singular otherwise; $\mathrm{cof}(\alpha)$ is always regular. $P(A)$ is the power set of A, i.e., the set of all subsets of A. $P_\kappa(A)$ is the set of all subsets of A of cardinality less than κ. Note that both $P(A)$ and $P_\kappa(A)$ are partially ordered by the inclusion relation.

We assume to a very considerable extent that the reader has some experience with forcing techniques. (For reference see either [6] or [20].) We differ from the usual practice by understanding the partial order on the forcing notion to mean that a larger condition gives more information about the eventual generic extension of the universe. The ground model is always denoted by V and we are not committing ourselves to assume that V is the whole universe and the extension is a Boolean valued model or to assume that V is a countable model and the extension is a two-valued model. The extension of V that we get by forcing with $\mathcal{P}$ is always denoted by $V[G]$ where $G \subseteq \mathcal{P}$ is a V generic filter over $\mathcal{P}$. Every element of $V[G]$ is a realization by G of some name of it that lies in V. (In the Boolean valued version a name of a is any representative of the equivalence class of a. In the two-valued version of [20] a name of $a \in V[G]$ is any $c \in V$ such that $K_G(c) = a$ where the function K_G is defined in [20].) The forcing language is the language of set theory extended by the introduction of a constant for every name in V. For every element a of V, we have the standard name of a, $\check{a}$, which always realized as a. As a general practice we shall not distinguish between the name of an element of $V[G]$ and the element it denotes. The forcing relation, denoted by $\Vdash$, is weak forcing.

If π is a forcing condition and Φ a statement in the forcing language of the form $\exists x\ (\Psi(x) \wedge x$ is an ordinal) then we say that π decides Φ if $\pi \Vdash \Psi(\check{\alpha})$ for some ordinal α.

If A is a set of ordinals such that, $A \in V[G]$ then $V[A]$ is the minimal model M of ZFC such that $V \subseteq M \subseteq V[G]$ and $A \in M$. If Γ is an automorphism of the forcing set (as a partially ordered set) then Γ naturally extends to the class of all names (see [6] or [20]), so that:

$$\pi \Vdash \Phi(a_1, \cdots, a_n) \quad \text{if and only if} \quad \Gamma\pi \Vdash \Phi(\Gamma a_1, \cdots, \Gamma a_n).$$

Moreover, if A is a set of ordinals in $V[G]$, which has a name which is invariant under Γ, then every element of $V[A]$ has a name which is invariant under Γ. Actually given a name for A, we can find for each element of $V[A]$ a name which is invariant under every automorphism which preserves the given name for A.

Given a set A, an ultrafilter U on $P_\kappa(A)$ is called normal if

(a) U is κ complete (i.e., U is closed under intersections of less than κ members);

(b) For every $a \in A$, the set $\{P \in P_\kappa(A) \mid a \in P\} \in U$;

(c) Every choice function on $P_\kappa(A)$ is *almost constant* with respect to U, i.e., if f is a function on $P_\kappa(A)$ such that for $P \neq \emptyset$ $f(P) \in P$ then there exists an $a \in A$ such that $\{P \mid f(P) = a\} \in U$.

κ is *A-supercompact* if there exists a normal ultrafilter on $P_\kappa(A)$. κ is *supercompact* if it is *A-supercompact* for every A. Note that if κ is *A-supercompact* for some A such that $\kappa \leqq |A|$ then κ is measurable. The definitions of a normal ultrafilter on $P_\kappa(A)$ and on supercompact cardinals were introduced in [17]. See also [11] since [17] is still unpublished.

For $P \in P_\kappa(\lambda)$, and α an ordinal $\alpha \leqq \lambda$ define $\alpha(P) = \mathrm{otp}(P \cap \alpha)$. As shown in [17] (see also [11]), if one forms the ultrapower $V^{P_\kappa(\lambda)}/U$ where U is a normal ultrafilter on $P_\kappa(\lambda)$, then it is well founded and for $\alpha \leqq \lambda$ the ordinal α is represented in the ultrapower by $P \to \alpha(P)$. From these remarks there easily follows:

LEMMA 1.1. *Let U be a normal ultrafilter on $P_\kappa(\lambda)$ where λ is some ordinal $\kappa \leqq \lambda$. Then*

(a) $\{P \in P_\kappa(\lambda) \mid \kappa(P) = P \cap \kappa$ *and it is an inaccessible cardinal*$\} \in U$.

(b) *If $\alpha \leqq \lambda$ is a cardinal (a regular cardinal) then* $\{P \in P_\kappa(\lambda) \mid \alpha(P)$ *is a cardinal (a regular cardinal)*$\} \in U$.

(c) *If $\gamma < \kappa$ and $\alpha, \beta \leqq \lambda$ $\alpha^{+\gamma} = \beta$ then* $\{P \in P_\kappa(\lambda) \mid \alpha(P)^{+\gamma} = \beta(P)\} \in U$.

(d) *If $\alpha, \beta \leqq \lambda$ and either* (i) $2^\alpha = \beta$ *or* (ii) $2^\alpha < \beta$ *or* (iii) $2^\alpha > \beta$ *then respectively*

(i) $\{P \in P_\kappa(\lambda) \mid 2^{\alpha(P)} = \beta(P)\} \in U$,

(ii) $\{P \in P_\kappa(\lambda) \mid 2^{\alpha(P)} > \beta(P)\} \in U$,

(iii) $\{P \in P_\kappa(\lambda) \mid 2^{\alpha(P)} < \beta(P)\} \in U$.

If P, Q are in $P_\kappa(\lambda)$ and $P \subseteq Q$ then P is *strongly included in Q*, or Q *strongly includes P* ($P \subsetneq Q$) if $P \subseteq Q$ and $\mathrm{otp}(P) < \mathrm{otp}(Q \cap \kappa)$. Note that this relation depends on κ, but we shall not mention κ since it will be constant in most applications. If U is a normal ultrafilter on $P_\kappa(\lambda)$, then condition (c) in the definition of normality implies a seemingly stronger condition:

Let F be a function from $P_\kappa(\lambda)$ into $P_\kappa(\lambda)$ such that for all $P \neq 0$ $F(P) \subsetneq P$ then F is constant on a set in U. By arguments similar to those given in [8] one can show that if for every $Q \in P_\kappa(\lambda)$, $A_Q \in U$ then $\{P \mid P \in \bigcap_{Q \subsetneq P} A_Q\} \in U$. (This last set is called the *diagonal intersection* of the system $\{A_Q \mid Q \in P_\kappa(\lambda)\}$.)

The notion of a normal ultrafilter on $P_\kappa(\lambda)$ generalizes the notion of normal ultrafilter on κ (see [8]) and like it, it can have nice partition properties, (Menas [15]). Let $B \subseteq P(A)$, then $[B]^{[n]}$ is the set of all n elements of subsets of B, which are totally ordered by inclusion, $[B]^{<\omega}$ is $\bigcup_{n<\omega}[B]^{[n]}$.

THEOREM 1.2. ([15]) *Let κ be A supercompact, then there exists a normal measure U on $P_\kappa(A)$ such that if F is a partition of $[P_\kappa(A)]^{<\omega}$ into some set of cardinality less than κ then there exists a set B, $B \in U$ such that for every $n < \omega$, F is constant on $[B]^{[n]}$.*

Actually, for the sets A we shall deal with, it can be shown that every normal ultrafilter on $P_\kappa(A)$ has the partition property described in Theorem 1.2.

Given two regular cardinals α and β, the standard forcing notion for collapsing β to α^+ is $\mathrm{Col}(\alpha,\beta)$ which is the set of all functions f whose domain is a subset of $\alpha \times \{\gamma \mid$ is a cardinal, $\alpha < \gamma < \beta\}$ such that for $(\gamma,\rho) \in \mathrm{Domain}(f)$, $f(\gamma,\rho) \in \rho$. And the cardinality of f is less than α. $\mathrm{Col}(\alpha,\beta)$ is partially ordered by inclusion and it is closed under unions of increasing sequences of length less than α. The cardinality of $\mathrm{Col}(\alpha,\beta)$ is clearly $\beta^{\underline{\alpha}}$. If $\gamma^{\underline{\alpha}} < \beta$ for every $\gamma < \beta$ then $\mathrm{Col}(\alpha,\beta)$ satisfies the β chain condition, i.e., every set of mutually incompatible elements of $\mathrm{Col}(\alpha,\beta)$ has cardinality less than β. Note that if $\alpha < \gamma < \beta$ then $\mathrm{Col}(\alpha,\gamma) \subseteq \mathrm{Col}(\alpha,\beta)$ and if $f \in \mathrm{Col}(\alpha,\beta)$ and the $\mathrm{Domain}(f) \subseteq \alpha \times \delta$ then $f \in \mathrm{Col}(\alpha,\delta)$. Hence if β is a limit cardinal, $f \in \mathrm{Col}(\alpha,\beta)$ then $f \in \mathrm{Col}(\alpha,\delta)$ for some $\delta < \beta$.

The intuitive motivation for $\mathrm{Col}(\alpha,\beta)$ is that $f(\delta,\gamma)$ for fixed γ is a partial information about a function which maps α onto γ. It is well known (see [6] and also [23] where $\mathrm{Col}(\omega,\kappa)$ is used) that using $\mathrm{Col}(\alpha,\beta)$ as a forcing notion generates a model in which $\beta \leqq \alpha^+$ and there are no new γ sequences of ordinals for $\gamma < \alpha$. If $\gamma^\alpha < \beta$ for all $\gamma < \beta$ then β is still a cardinal in the extension, hence $\beta = \alpha^+$.

§2. The forcing conditions

As described in the introduction we start by proving Theorem 1 (a). We are given the natural number k and we want to construct a model in which $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} = \aleph_{\omega+k}$. We assume that we are given a ground model, V, in which there is a cardinal κ such that:

(a) $2^\kappa = \kappa^{+k}$,

(b) κ is $\kappa^{+(k-1)}$ supercompact.

It follows from [21] (see also [14], Ch. 4) that this is consistent with the existence of a supercompact cardinal. Actually it is enough to start from the assumption that there exists κ such that κ is κ^{+k} supercompact.

Our forcing notion, that we define in this section, will be a generalization of that of Prikry [16] and it changes the cofinality of each one of $\kappa, \kappa^{+1}, \cdots, \kappa^{+(k-1)}$ to ω simultaneously (hence collapsing $\kappa^{+1}, \kappa^{+2}, \cdots, \kappa^{+(k-1)}$ to κ). We shall keep denoting these ordinals by κ^{+i}, $i = 1, \cdots, k-1$ though κ^{+i} in the sense of $V[G]$ is of course different. Since κ will become also $\aleph_\omega$ in the sense of $V[G]$ we shall denote κ^{+i} in the sense of $V[G]$ by $\aleph_{\omega+i}$.

Besides this change of cofinality, many cardinals less than κ will be collapsed such that in the extension κ will become $\aleph_\omega$, while preserving its strong limit status. On the other hand we are able to show that κ^{+k} is not collapsed. $V[G]$ is not the model we are looking for since in it $2^{\aleph_\omega} = \aleph_{\omega+1}$. The model we are looking for, which is a witness for the truth of Theorem 1 (a), is a certain submodel of $V[G]$ to be described in the next section. The present section is devoted to the construction of $V[G]$.

Let U be a fixed normal ultrafilter on $P_\kappa(\kappa^{+(k-1)})$ which satisfies the conclusions of Theorem 1.2. (In this case any normal ultrafilter will do.) Let D be the set $\{P \in P_\kappa(\kappa^{+(k-1)}) \mid P \cap \kappa$ is an ordinal which is an inaccessible cardinal and $(\kappa^{+i}(P))^+ = \kappa^{+i+1}(P)$ for $0 \leqq i \leqq k-2\}$. By Lemma 1.1 $D \in U$.

Definition 2.1. $\mathcal{P}$ (the forcing notion we shall use) is the set of all finite sequences of the form $\pi = \langle P_1, \cdots, P_l, f_0, \cdots, f_l, A, G\rangle$ where

(a) $P_i \in D$, for $1 \leqq i \leqq l$, $P_i \subsetneqq P_{i+1}$ for $1 \leqq i \leqq l-1$,

(b) if we define $\kappa_i = \operatorname{otp}(P_i \cap \kappa)$ then $f_0 \in \operatorname{Col}(\omega_1, \kappa_1)$, $f_i \in \operatorname{Col}(\kappa_i^{+k}, \kappa_{i+1})$ for $1 \leqq i \leqq l-1$ and $f_l \in \operatorname{Col}(\kappa_l^{+k}, \kappa)$ (note that since κ is a limit cardinal $f_l \in \operatorname{Col}(\kappa_l^{+k}, \beta)$ for some $\beta < \kappa$),

(c) $A \subseteq D$, $A \in U$ and for every $Q \in A$, $P_l \subsetneqq Q$ and $f_l \in \operatorname{Col}(\kappa_l^{+k}, Q \cap \kappa)$,

(d) G is a function defined on A such that for $Q \in A$ $G(Q) \in \operatorname{Col}((\kappa \cap Q)^{+k}, \kappa)$ and if $P \in A$, $P \subsetneqq Q$ then $G(P) \in \operatorname{Col}((\kappa \cap P)^{+k}, \kappa \cap Q)$. l is called the length of the condition π, $\langle P_1, \cdots, P_l\rangle$ its "P part" and $\langle f_1, \cdots, f_l\rangle$ its "f part".

It seems appropriate at this point to give some intuitive motivation for the definition of the forcing conditions. Since we want to change the cofinality of every κ^{+i} to ω for $0 \leqq i < k$ we need to introduce a cofinal ω sequence for each of them. Given the condition π as in Definition 2.1, the finite sequence whose j-th member is $\operatorname{Sup}(P_j \cap \kappa^{+i})$ is an approximation to the sequence cofinal in κ^{+i}. A is the set of possible candidates for extending the sequence $P_1, \cdots, P_l$, i.e., all future members will be picked from A. The assumption $A \in U$ guarantees that this set of candidates is large enough. Since we want to transform κ to $\aleph_\omega$ we have to collapse many cardinals below κ, and our forcing conditions are supposed to collapse every cardinal which is strictly between ω_1 and κ_1 to ω_1. f_0 is a partial information about such a collapse. Similarly we collapse every cardinal strictly between κ_1^{+k} and κ_2 to κ_1^{+k}. f_1 is a partial information about this collapse. Note that we are leaving $\kappa_1, \kappa_1^+, \cdots, \kappa_1^{+(k-1)}$ with collapsing. The reasons should be clear from the proof of the main technical lemma which is Theorem 2.6. The role played by $f_2, \cdots, f_{l-1}$ is the analogue of f_1 for κ_2^{+k}, κ_3; κ_3^{+k}, κ_4; $\cdots$ respectively. f_l is supposed to be a partial information about the collapse of the cardinals between κ_l^{+k} and κ_{l+1}. But we do not know yet what is κ_{l+1}, hence we assume it is in $\operatorname{Col}(\kappa_l^{+k}, \kappa)$ and the conditions imposed on A guarantee that no matter which element of A is picked as P_{l+1}, $f_{l+1} \in \operatorname{Col}(\kappa_l^{+k}, \kappa_{l+1})$.

By G we are making another commitment. $G(P)$ will be of significance just in case we decide at some future condition to use P as P_n for some $l < n$. In that case f_n will have to extend $G(P)$. Hopefully these remarks should be a sufficient motivation for the following definition of the partial order on $\mathcal{P}$.

DEFINITION 2.2. Let $\pi, \pi' \in \mathcal{P}$, $\pi = \langle P_1, \cdots, P_n, f_0, \cdots, f_n, A, G\rangle$, $\pi' = \langle Q_1, \cdots, Q_l, g_0, \cdots, g_l, B, H\rangle$. We say that π' extends π ($\pi \leqq \pi'$) if

(a) $n \leqq l$ and $Q_i = P_i$ for $1 \leqq i \leqq n$,

(b) $f_i \subseteq g_i$ for $0 \leqq i \leqq n$,

(c) $Q_i \in A$ and $G(Q_i) \subseteq g_i$ for $n < i \leqq l$,

(d) $B \subseteq A$,

(f) for every $P \in B$, $G(P) \subseteq H(P)$.

DEFINITION 2.3. Let π and π' be as in Definition 2.2 $\pi \leqq \pi'$. Let $0 \leqq j \leqq n$, then π' is called a j-direct extension of π if

(a) $f_i = g_i$ for $j \leqq i \leqq n$,

(b) $G(Q_i) = g_i$ for $n < i \leqq l$,

(c) $B = \{P \mid P \in A,\ Q_l \subsetneq P\}$,

(d) for $P \in B$, $G(P) = H(P)$.

π' is a direct extension of π if it is a 0-direct extension of π.

Note that if π' is a j-direct extension of π then it is uniquely determined by $g_0, \cdots, g_{j-1}$ and $Q_{n+1}, \cdots, Q_l$, hence we call π' the j-direct extension of π determined by $\langle g_0, \cdots, g_{j-1}\rangle$ and $\langle Q_{n+1}, \cdots, Q_l\rangle$. (In case $j = 0$ omit mentioning j and $\langle g_0, \cdots, g_{j-1}\rangle$). The dual of a j-direct extension is given by the following definition:

DEFINITION 2.4. Let π and π' be as in Definition 2.2 $\pi \leqq \pi'$, $0 \leqq j \leqq n$, then π' is called a j-length preserving extension of π if

(a) $n = l$,

(b) $f_i = g_i$ for $0 \leqq i < j$.

Note that if $\pi \leqq \pi'$ then there is a unique π'' such that

(a) $\pi \leqq \pi'' \leqq \pi'$,

(b) π'' is a j-direct extension of π,

(c) π' is a j-length preserving extension of π''.

We call this unique π'' the *j-interpolant of π and π'*. Note that if $\pi \leqq \pi' \leqq \pi''$ then the j-interpolant of π' and π'' extends the j-interpolant of π and π''. Also note that if π'' is the j-interpolant of π and π', then the j-interpolant of π'' and π' is π'' itself. *π restricted to j* is $\langle P_1, \cdots, P_j, f_0, \cdots, f_{j-1}\rangle$, denoted by $\pi \restriction j$.

LEMMA 2.5. *Let π, j be as in Definition 2.4 $\langle \pi_\gamma \mid \gamma < \lambda \rangle$ is a sequence of j-length preserving extensions of π where for $\beta \leqq \gamma$, $\pi_\beta \leqq \pi_\gamma$ and $\lambda \leqq \kappa_j^{+(k-1)}$ if $j > 0$ and $\lambda \leqq \omega$ if $j = 0$. Then there exists π' which is a j-length preserving extension of each $\langle \pi_\gamma \mid \gamma < \lambda \rangle$ and of π.*

PROOF. Let $\pi = \langle P_1, \cdots, P_n, f_0, \cdots, A, G \rangle$ and $\gamma < \lambda$, $\pi_\gamma = \langle P_1, \cdots, P_n, f_0, \cdots, f_{j-1}, f_j^\gamma, \cdots, A^\gamma, G^\gamma \rangle$. Define π' by $\langle P_1, \cdots, P_n, f_0, \cdots, f_{j-1}, g_j, \cdots, g_n, B, H \rangle$ where $g_i = \bigcup_{\gamma<\lambda} f_i^\gamma$, $j \leqq i \leqq n$ (here we used $\lambda \leqq \kappa_j^{+(k-1)}$ since $\mathrm{Col}(\kappa_i^{+k}, \kappa_{i+1})$ is closed under increasing unions of length at most $\kappa_i^{+(k-1)}$, $B = \bigcap_{\gamma<\lambda} A^\gamma$ and $H(P) = \bigcup_{\gamma<\lambda} G^\gamma(P)$ for $P \in B$. (Here we used the κ completeness of U, for inferring $B \in U$.) Clearly π' is the required condition.

The main technical device that we shall use is the following theorem:

THEOREM 2.6. *Let π be a condition of length n and $j \leqq n$. Φ is a statement of the form $\exists x$ (x is an ordinal $\wedge \Psi(x)$) in the forcing language for $\mathcal{P}$ then there is a j-length preserving extension π, π' which decides Φ up to j-direct extensions, i.e., if $\pi' \leqq \pi''$ and π'' decides Φ, then the j-interpolant of π' and π'' decides Φ.*

PROOF. We shall break the proof into two lemmas. The first lemma will follow from the second. The theorem follows immediately from the first lemma.

LEMMA 2.7. *Let Φ and π be as in Theorem 2.6. Let η be the restriction to j of some extension of π. Then there exists a j-length preserving extension π' of π for which Theorem 2.6 holds for every π'' satisfying $\pi \restriction j = \eta$.*

PROOF OF THEOREM 2.6 FROM LEMMA 2.7. It can be easily verified that the set $\{\eta \mid \eta$ is a restriction to j of some extension of $\pi\}$ has cardinality κ_j, because it is easily seen to have the same cardinality as $\mathrm{Col}(\omega_1, \kappa_1) \times \mathrm{Col}(\kappa_1^{+k}, \kappa_2) \times, \cdots, \times \mathrm{Col}(\kappa_{j-1}^{+k}, \kappa_j)$. All of $\{\kappa_i \mid 1 \leqq i \leqq j\}$ are inaccessible and $\mathrm{Col}(\alpha, \beta)$ for inaccessible β has cardinality β, hence the cardinality is $\kappa_1 \times \kappa_2, \cdots, \times \kappa_j = \kappa_j$.

Let $\langle \eta_\gamma \mid \gamma < \kappa_j \rangle$ be an enumeration of this set. By induction construct an increasing sequence of conditions $\langle \pi_\gamma \mid \gamma < \kappa_j \rangle$ each of which is a j-length preserving extension of π. $\pi_0 = \pi$; for a limit ordinal π_γ is any j-length preserving extension of π which is also an extension of π_β for $\beta < \gamma$. (π_γ exists by Lemma 2.5.) For $\gamma = \delta + 1$, $\pi_{\delta+1}$ is a j-length preserving extension of π_δ (hence of π) which satisfies the conclusion of Lemma 2.7, where we replace η by η_δ and π by $\pi_{\delta+1}$ in the statement of the lemma.

Let π' be a j-length preserving extension of π which is also an extension of π_β for $\beta < \kappa_j$. (Again use Lemma 2.5.) π' satisfies the requirements of the theorem because if $\pi' \leqq \pi''$ and π'' decides Φ (let us say, $\pi \Vdash \Psi(\check{\beta})$) then let $\eta_\delta = \pi \restriction j$ for some $\delta < \kappa_j$. By our construction $\pi_{\delta+1} \leqq \pi''$, hence by the definition of $\pi_{\delta+1}$ the j-interpolant of $\pi_{\delta+1}$ and π forces $\Psi(\check{\alpha})$ for some α, but since the j-interpolant of π' and π'' extends the j-interpolant of $\pi_{\delta+1}$ and π'', we get that the j-interpolant of π' and π'' forces $\Psi(\check{\alpha})$. □

We still have to pay the debt of proving Lemma 2.7, for which we need:

LEMMA 2.8. *Let Φ, π and η be as in Lemma 2.7 where the length of π is n and l is a fixed natural number, then there is a j-length preserving extension of π which satisfies the conclusions of Lemma 2.7 for every π'' of length $n + l$.*

PROOF OF LEMMA 2.7 FROM LEMMA 2.8. By induction on $l < \omega$ we construct an increasing sequence $\langle \pi_l \mid l < \omega \rangle$ of j-length preserving extensions of π. π_0 is π and π_{l+1} is a j-length preserving extension of π_l which satisfies the conclusions of Lemma 2.8 where we replace π by π_l. Let π' be a j-length preserving extension of π which is an extension of π_l for every $l < \omega$. (Again Lemma 2.5 is used.) π' satisfies the requirements of Lemma 2.7 because if $\pi' \leqq \pi''$, $\pi'' \Vdash \Psi(\check{\beta})$ for some β where the length of π'' is $n + l$ for some $l < \omega$, then $\pi_{l+1} \leqq \pi''$ and by the definition of π_{l+1} the j-interpolant of π_{l+1} and π'' forces $\Psi(\check{\alpha})$ for some α, but the interpolant of π' and π'' extends this last interpolant hence the j-interpolant of π' and π'' forces $\Psi(\check{\alpha})$. □

PROOF OF LEMMA 2.8. The proof is by induction on l. Let $\pi = \langle P_1, \cdots, P_n, f_0, \cdots, f_n, A, G \rangle$, $l = 0$. Distinguish two cases:

Case I. There is π'', $\pi \leqq \pi''$ which is of length n such that $\pi \restriction j = \eta$ and π'' decides Φ. Fix such π'' where $\pi'' = \langle P_1, \cdots, P_n, g_0, \cdots, g_n, B, H \rangle$. Let $\pi' = \langle P_1, \cdots, P_n, f_0, \cdots, f_{j-1}, g_j, \cdots, g_n, B, H \rangle$. Then π' is the required condition which satisfies the conclusion of Lemma 2.8 for $l = 0$, since every extension of π', whose restriction to j is η, is an extension of π''.

Case II. Case I fails, let $\pi' = \pi$. The lemma is vacuously true, since there is no extension of π of length $n + 0$ which decides Φ, and whose restriction to j is η.

The induction step. Assume that the lemma holds for some fixed l and for all n and the condition π of length n. And we shall prove it for $l + 1$. So given π of length n, $\pi = \langle P_0, \cdots, P_n, f_0, \cdots, f_n, A, G \rangle$. Since the partial order $\subsetneq$ on D is well founded ($P \subsetneq Q$ implies $\kappa \cap P < \kappa \cap Q$) we can extend it to a well ordering of D. Let $\lesssim$ be this well ordering. By induction on $\lesssim$ restricted to A, we define a sequence of conditions $\langle \pi_P \mid P \in A \rangle$ where π_Q has the form:

(I) $$\pi_Q = \langle P_1, \cdots, P_n, Q, f_0, \cdots, f_{j-1}, f_j^Q, \cdots, f_n^Q, f^Q, B^Q, H^Q \rangle,$$

(II) $$\pi \leqq \pi_Q,$$

(III) If $T, Q, S \in A$, $T \subsetneq S$, $Q \subsetneq S$ and $S \in B^T \cap B^Q$ then $H^T(S)$ and $H^Q(S)$ are compatible. (Actually if $T \lesssim Q$ then $H^T(S) \subseteq H^Q(S)$.)

Note that each π_P is an extension of π of length $n+1$ such that $\pi_P \restriction j = \pi \restriction j$. Assume we already defined π_Q for $Q \lesssim P$ such that (I), (II), (III) hold for every $Q, T \lesssim P$. Note that we picked the order $\lesssim$ such that if $Q \subsetneq P$, $Q \in A$, then $Q \lesssim P$, hence π_Q is already defined for all $Q \subsetneq P$.

Define $\chi_P = \langle P_1, \cdots, P_n, P, f_0, \cdots, f_n, g^P, A^P, G^P \rangle$ where $g^P = \cup \{H^Q(P) \mid Q \subsetneq P,\ Q \in A\}$ (in case $P \notin B^Q$ we understand $H^Q(P) = G(P)$), $A^P = \{T \mid T \in A \cap \{B^Q \mid Q \subsetneq T,\ Q \lesssim P\},\ P \subsetneq T\}$. Note that by the normality of U, $A^P \in U$ (we use the notes before Theorem 1.2);

$$G^P(T) = \cup \{H^Q(T) \mid Q \subseteq T, Q \leqslant T\} \quad \text{for} \quad T \in A^P\}.$$

χ_P is a condition: $g^P \in \mathrm{Col}((P \cap \kappa)^{+k}, \kappa)$ since each $H^Q(P)$ is in $\mathrm{Col}((P \cap \kappa)^{+k}, \kappa)$, any two of them are compatible, by the induction hypothesis (III), and the cardinality of $\{H^Q(P) \mid Q \subsetneq P,\ Q \in A\}$ is at most the cardinality of $\{Q \mid Q \subsetneq P\}$ which is $|P|^{\underline{P \cap \kappa}}$. Since $|P| = (P \cap \kappa)^{+(k-1)}$ and $P \cap \kappa$ is inaccessible, $|P|^{\underline{P \cap \kappa}}$ can easily be seen to be $|P| = (P \cap \kappa)^{+(k-1)}$. Thus we proved that $g^P \in \mathrm{Col}((P \cap \kappa)^{+k}, \kappa)$ since $\mathrm{Col}((P \cap \kappa)^{+k}, \kappa)$ is closed under unions of less than $(P \cap \kappa)^{+k}$ compatible elements. (By the way, this is the point where we use the fact that we start collapsing from κ_i^{+k}.) Similar argument shows that $G^P(T) \in \mathrm{Col}((T \cap \kappa)^{+k}, \kappa)$. The other clauses in Definition 2.1 can also be verified to hold for χ_P.

Now apply the present lemma for l, where instead of π we have χ_P and instead of n, $n+1$. (The induction hypothesis is that the lemma holds for l and *any* condition π.) We get $\chi_P \leqq \pi'$ which satisfies the conclusion of the lemma for l, χ_P and $n+1$. Define π_P to be this π'. (I), (II), (III) are not violated by this definition of π_P: (I) because π_P is a j-length preserving extension of χ_P, (II) because $\pi \leqq \chi_P$, and (III) because of the particular way in which we defined $G^P(T)$. (Of course $G^P(T) \subseteq H^P(T)$ for every $T \in B^P$.) This completes the inductive definition of the sequence $\langle \pi_P \mid P \in A \rangle$.

We are now ready for the definition of the condition π' which will be a witness for the lemma for π, $l+1$, and n. The first step is to observe that there exists a set $B \in U$ $(B \subseteq A)$ such that for $P \in B$, $f_j^P, \cdots, f_n^P$ are constant. For $j \leqq i \leqq n$, $f_i^P \in \mathrm{Col}(\kappa_i^{+k}, \kappa_{i+1})$ and $|\mathrm{Col}(\kappa_i^{+k}, \kappa_{i+1})| = \kappa_{i+1}^{\kappa_{i+1}} = \kappa_{i+1} < \kappa$, hence by the κ completeness of U we can find $B_1 \subseteq A$, $B_1 \in U$, for which $f^P_j, \cdots, f^P_{n-1}$ are constant. For all $P \in A$ $f^P_n \in \mathrm{Col}(\kappa^{+k}_n, \kappa \cap P)$, therefore since $\kappa \cap P$ is inaccessible $f^P_n \in \mathrm{Col}(\kappa^{+k}_n, \alpha_P)$ for some $\alpha_P < \kappa \cap P$. (Hence $\alpha_P \in P \cap \kappa$.) By normality of U there is a set $B_2 \in U$, $B_2 \subseteq B_1$ such that on B_2, α_P is constantly equal to some $\alpha < \kappa$. Since the cardinality of $\mathrm{Col}(\kappa^{+k}_n, \alpha)$ is less than κ, again we can involve the κ completeness of U, and get $B \subseteq B_2$, $B \in U$ such that for $P \in B$, f^P_n is constant. Let $g_j, \cdots, g_n$ be the constant value of $f^P_j, \cdots, f^P_n$ for $P \in B$.

π' is defined to be $\langle P_1, \cdots, P_n,\ f_0, \cdots, f_{j-1},\ g_j, \cdots, g_n, C, H \rangle$ where $C = B \cap \{P \mid P \in \cap\{B^Q \mid Q \in A,\ Q \subsetneq P\}\}$ and $H(P) = f^P$ for $P \in C$.

Note that by our definition of f^P

$$\text{(IV)} \qquad \{H^Q(P) \mid Q \in A,\ Q \subsetneq P\} \subseteq f^P \quad \text{and} \quad G(P) \subseteq f^P.$$

We claim that π' satisfies the lemma for $l+1$. Clearly $\pi \leqq \pi'$ (note that for $j \leqq i \leqq n$, $f_i \subseteq g_i$). And it is a j-length preserving extension of π. Let $\pi' \leqq \pi''$, π'' of length $n + (l+1)$, $\pi \restriction j = \eta$ and π'' decides Φ (i.e. $\pi'' \Vdash \Psi(\check{\alpha})$ for some α). Let χ be the j-interpolant of π' and π'', hence χ is the j-direct extension of π' determined by $\langle h_0, \cdots, h_{j-1}\rangle$ and $\langle P, Q_1, \cdots, Q_l\rangle$ for some $P, Q_1, \cdots, Q_l \in C$. Note that since $\pi \restriction j = \eta$, $\langle P_1, \cdots, P_j,\ h_0, \cdots, h_{j-1}\rangle = \eta$. By our construction of π', $\pi_P \leqq \chi \leqq \pi''$ because $Q_1, \cdots, Q_l \in C$ and $P \subsetneq Q_1 \subsetneq Q_2 \subsetneq Q_l$ implies $Q_1, \cdots, Q_l \in B^P$, $H^P(Q) \subseteq H(Q)$ by (IV), $f^P_i = g_i$ for $j \leqq i \leqq n$, and $H(P) = f^P$.

Now use the definition of π_P, $\pi_P \leqq \pi''$, $\pi \restriction j = \eta$, π'' decides Φ and π'' has length $(n+1)+l$, therefore the j-interpolant of π_P and π'' (denote it by χ') decides Φ. But the j-interpolant of χ and π'' (which is χ itself) extends the j-interpolant of x_P and π'' which is χ', hence $\chi' \leqq \chi$ and χ decides Φ. □

Corollary 2.9. *Let π be a condition of length n, $j \leqq n$, $\mu \leqq \kappa_j^{+(k-1)}$ and b be a name which is forced by π to be a function with domain μ and whose values are ordinals. Then there is an extension π' of π such that for all $\pi' \leqq \pi''$ and $\lambda < \mu$ if $\pi'' \Vdash b(\check{\lambda}) = \check{\beta}$ for some β, then the j-interpolant of π' and π'' forces $b(\check{\lambda}) = \check{\beta}$.*

Note. Intuitively this corollary implies that every function from $\mu \leqq \kappa_j^{+(k-1)}$ into ordinals is generated by the "P part" of the generic filter and by the first j coordinates of the "F part" because given the function named b, we can find in the generic filter a condition π' for which the corollary holds, and if we want to get information about the value of $b(\lambda)$ we just have to know the j-direct extensions of π' which lies in the generic filter. But these j-direct extensions are determined by the "P part" and the first j-coordinates of the "F part".

Proof of Corollary 2.9. By Theorem 2.6 and Lemma 2.5 construct an increasing sequence of j-length preserving extensions of π $\langle \pi_\lambda \mid \lambda < \mu \rangle$ where

$\pi_0 = \pi$, π_λ for limit λ is any j-length preserving extension of $\langle \pi_\gamma \mid \gamma < \lambda \rangle$ (Lemma 2.5) and $\pi_{\lambda+1}$ is a j-length preserving extension which satisfies Theorem 2.8 with respect to π_λ, and the statement $\exists x\ (b(\check{\lambda}) = x)$. Let π' be a common extension of all the members of $\langle \pi_\lambda \mid \lambda < \mu \rangle$ by Lemma 2.5 (note that $\mu \leqq \kappa_j^{+(k-1)}$). π' is the required condition because if $\pi' \leqq \pi''$, $\pi'' \Vdash b(\check{\lambda}) = \check{\beta}$ for some β. $\pi_\lambda \leqq \pi' \leqq \pi''$, hence the j-interpolant of π_λ and π'', χ, forces $b(\check{\lambda}) = \check{\gamma}$ for some γ, but we must have $\gamma = \beta$ because $\pi \leqq \pi_\lambda \leqq \chi \leqq \pi''$ and π forces b to be a function. If χ' is the j-interpolant of π' and π'' we have $\chi \leqq \chi'$, hence $\chi' \Vdash b(\check{\lambda}) = \check{\beta}$. □

§3. The model

In this section we verify that a certain submodel of $V[G]$, where G was V generic over $\mathscr{P}$, is the right model which yields the consistency result of Theorem 1 (a). Before we describe the submodel we need some information about $V[G]$.

If G is V generic over $\mathscr{P}$, then clearly for every n we have a condition in G of length n. We define the "P part" of G to be the union of all "P parts" of members of G and it is an infinite sequence $\langle P_n \mid n < \omega, n \neq 0 \rangle$. Similarly we have the sequence $\langle \kappa_n = P_n \cap \kappa \mid n < \omega,\ n \neq 0 \rangle$. For every $n < \omega$ let

$$F_n = \cup \{ f \mid \text{Some } \pi \text{ in } G \text{ has the form } \pi = \langle P_1, \cdots, P_l, f_1, \cdots, f_l, A, H \rangle \text{ with } n \leqq l \text{ where } f_n = f \}.$$

F_n is clearly a function which by standard forcing arguments (see [6], Model VI) can be shown to have domain $\kappa_n^{+k} \times \{ \gamma \mid \kappa_n^{+k} < \gamma < \kappa_{n+1},\ \gamma \text{ is a cardinal} \}$ and range κ_{n+1}. Moreover for every fixed cardinal γ, $\kappa_n^{+k} < \gamma < \kappa_{n+1}$, $F(\alpha, \gamma)$ as a function of α is a function from κ_n^{+k} *onto* γ. Hence in $V[G]$ no γ between κ_n^{+k} and κ_{n+1} is a cardinal. (Later we show that κ_n^{+k} and κ_{n+1} *are* cardinals in $V[G]$.) The sequence $\langle F_n \mid n < \omega \rangle$ is called the F part of G.

The next lemma gives some information about cardinals in $V[G]$ above κ.

LEMMA 3.1. *Let $\langle P_n \mid n < \omega \rangle$ be the P part of a V generic filter over P. Let $0 \leqq i \leqq k-1$, then* $\mathrm{Sup}(\{ P_n \cap \kappa^{+i} \mid n < \omega \})$ *is cofinal in κ^{+i}. In particular $\langle \kappa_n = P \cap \kappa^{+0} \mid n < \omega \rangle$ is cofinal in κ.*

PROOF. Let $\alpha \in \kappa^{+i}$ and let $\pi \in \mathscr{P}$. We show that π can be extended to a condition which forces for some $n < \omega$ that $\mathrm{Sup}(P_n \cap \kappa^{+i}) > \alpha$. If π is $\langle P_1, \cdots, P_l, f_0, \cdots, f_l, A, H \rangle$, pick P to be a member of A containing $\alpha + 1$, which exists since $A \in U$ and U is a normal ultrafilter over $P_\kappa(\kappa^{+(k-1)})$. Let π' be the direct extension of π determined by P. Clearly $\pi' \Vdash \mathrm{Sup}(P_{l+1} \cap \kappa^{+i}) > \alpha$. □

Hence in $V[G]$, κ^{+i} for $1 \leqq i < k$ has a cofinality ω therefore none of $\kappa^{+1}, \kappa^{+2}, \cdots, \kappa^{+(k-1)}$ can be a cardinal in $V[G]$.

Let G be V generic over $\mathcal{P}$. Let $G \upharpoonright j$ be the set

$$\{\langle f_0, \cdots, f_{j-1}\rangle \mid \text{Some } \pi \in G \text{ has the form } \langle P_1, \cdots, P_l, f_0, \cdots, f_l, A, H\rangle \text{ with } j \leqq l \text{ for some } f_{j+1}, \cdots, f_l, A, H\}.$$

The reader should have no difficulty in verifying that $G \upharpoonright j$ is V generic over $\mathcal{P}_j$ where $\mathcal{P}_j$ is $\mathrm{Col}(\omega_1, \kappa_1) \times \mathrm{Col}(\kappa_1^{+k}, \kappa_2) \times \cdots \times \mathrm{Col}(\kappa_{j-1}^{+k}, \kappa_j)$ which is a partially ordered set in the natural extension of the partial order on $\mathrm{Col}(\kappa_i^{+k}, \kappa_{i+1})$ (see [20], section 8, where similar arguments are used).

The next theorem shows that every bounded subset of κ in $V[G]$ is in $V[G \upharpoonright j]$ for $j < \omega$ large enough.

THEOREM 3.2. *Let $\pi \in \mathcal{P}$, b a name and $\pi \Vdash b \subseteq \mu$ where $\mu \leqq \kappa_j^{+(k-1)}$ for some j, and the length of π is at least j. Then $\pi \Vdash b \in V[G \upharpoonright j]$.*

PROOF. We show that every extension π' of π has an extension χ and a name, c, in the language of $\mathcal{P}_j$ such that $\chi \Vdash c = b$. (Note the c can be considered also to be a name for $\mathcal{P}$ whose realization is $K_{G \upharpoonright j}(c)$. If we know G, we know of course also $G \upharpoonright j$.)

Naturally we can consider b to be realized as a function from μ into $\{0,1\}$. Hence we can use Corollary 2.9 for π' instead of π, and get an extension $\pi' \leqq \chi'$ such that for every $\lambda \in \mu$ if $\chi' \leqq \chi''$ and $x'' \Vdash b(\lambda) = i$ for some $i \in \{0,1\}$ then the j-interpolant of χ' and χ'' forces the same statement.

Let $\chi' = \langle P_1, \cdots, P_n, f_0, \cdots, f_n, A, H\rangle$. We define a partition of $[A]^{<\omega}$ into $P(\mathcal{P}_j \times \mu \times 3)$. Note that the cardinality of $\mathcal{P}_j$ is κ_j, hence the cardinality of $P(\mathcal{P}_j \times \mu \times 3)$ is $2^{\kappa_j + \mu + 3}$ which is less than κ. The partition F is defined by: $F(\{Q_1, \cdots, Q_l\})$ is the set of all triples of the form (η, λ, i) where $\eta \in \mathcal{P}_j$, $\eta = \langle g_0, \cdots, g_{j-1}\rangle$ where $f_i \subseteq g_i$ for $0 \leqq i < j$, $\lambda \in \mu$ and $i \in 3$ such that

(a) if $i = 0$ the j-direct extension of χ' determined by η and $\langle Q_1, \cdots, Q_l\rangle$ forces $\check{\lambda} \notin b$ (which is the same as $b(\lambda) = 0$),

(b) if $i = 1$ the j-direct extension of χ' determined by η and $\langle Q_1, \cdots, Q_l\rangle$ forces $\check{\lambda} \in b$ (which is the same as $b(\lambda) = 1$),

(c) if $i = 2$ the j-direct extension of χ' determined by η and $\langle Q_1, \cdots, Q_l\rangle$ forces neither $\check{\lambda} \in b$ nor $\check{\lambda} \notin b$.

Now use Theorem 1.2 for the partition F and get a set B, $B \subseteq A$, $B \in U$ such that for $l < \omega$ F is constant on $[B]^l$. Let this constant be E_l. Note that if for some η, λ $(\eta, \lambda, 1) \in E_l$, then for no l' and no η' compatible with η can we have

$(\eta', \lambda, 0) \in E_{l'}$, otherwise if η'' is a common extension of η and η', and $m = \max(l, l')$, pick $Q_1 \subseteq, \cdots, Q_m$, $Q_i \in B$, then the j-direct extension of χ' determined by η'' and $\langle Q_1, \cdots, Q_m \rangle$ forces $\check{\lambda} \in b$ (since it is an extension of the j-direct extension determined by η and $\langle Q_1, \cdots, Q_l \rangle$ and $(\eta, \lambda, 1) \in E_l = F(\{Q_1, \cdots, Q_l\})$. On the other hand it forces $\check{\lambda} \notin b$ (since it is an extension of the j-direct extension determined by η' and $\langle Q_1, \cdots, Q_{l'} \rangle$ and $(\eta', \lambda, 0) \in E_{l'} = F(\{Q_1, \cdots, Q_{l'}\})$, which is a contradiction.

Define the condition χ to be $\langle P_1, \cdots, P_n, f_0, \cdots, f_n, B, H \restriction B \rangle$. We claim that χ forces $b \in V[G \restriction j]$. Let c be a name for $\mathcal{P}_j$ which is always realized as the set (where $\tilde{G}$ is V generic over $\mathcal{P}_j$ such that $\langle f_0, \cdots, f_{j-1} \rangle \in \tilde{G}$)

(I) $$\{\lambda \mid \lambda < \mu, \text{ for some } l < \omega \text{ and } \eta \in \tilde{G} \ (\eta, \lambda, 1) \in E_l\}.$$

Note that if $\tilde{G}$ is V generic over $\mathcal{P}_j$ and $\lambda < \mu$ then either $(\eta, \lambda, 1) \in E_l$ for some $\eta \in \tilde{G}$ and l, or $(\eta, \lambda, 0) \in E_l$ for some $\eta \in \tilde{G}$ and l (by the remarks above we cannot have both cases simultaneously), because if $\tilde{\eta} \in \mathcal{P}_j$, $\langle f_0, \cdots, f_{j-1} \rangle \leqq \eta$, consider the condition in $\mathcal{P}$

$$\langle P_1, \cdots, P_n, g_0, \cdots, g_{j-1}, f_j, \cdots, f_n, B, H \restriction B \rangle,$$

where $\tilde{\eta} = \langle g_0, \cdots, g_{j-1} \rangle$ and pick an extension of it which either forces $\check{\lambda} \in b$ or forces $\check{\lambda} \notin b$. Since this condition is an extension of $\chi' \leqq \chi$ we have by definition of χ' $Q_1, \cdots, Q_l \in B$ and $\tilde{\eta} \leqq \tilde{\tilde{\eta}}$ such that the j-direct extension of χ' determined by $\langle Q_1, \cdots, Q_l \rangle$ and $\tilde{\tilde{\eta}}$ either forces $\lambda \in \check{b}$ or it forces $\check{\lambda} \notin b$. Hence either $\langle \tilde{\tilde{\eta}}, \lambda, 1 \rangle \in E_l$ or $\langle \tilde{\tilde{\eta}}, \lambda, 0 \rangle \in E_l$. Therefore since $\tilde{\eta}$ was any member of $\mathcal{P}_j$ extending $\langle f_0, \cdots, f_{j-1} \rangle$, $\tilde{G}$ must contain $\tilde{\tilde{\eta}}$ such that for some l either $\langle \eta, \lambda, 1 \rangle \in E_l$ or $\langle \eta, \lambda, 0 \rangle \in E_l$. What this leads to is that an alternative definition of c can be the name whose realization is

(II) $$\mu - \{\lambda \mid \lambda < \mu, \text{for some } l < \omega \text{ and } \eta \in \tilde{b}, (\eta, \lambda, 0) \in E_l\}.$$

χ forces that the realization of b by G is the same as the realization of c by $G \restriction j$. Assume otherwise, then there exists $\chi \leqq \bar{\chi}$ and $\lambda < \mu$ such that either

$$\bar{\chi} \Vdash \check{\lambda} \in b \wedge \check{\lambda} \notin \text{"The realization of } c \text{ by } G \restriction j\text{"}$$

or

$$\bar{\chi} \Vdash \check{\lambda} \notin b \wedge \check{\lambda} \in \text{"The realization of } c \text{ by } G \restriction j\text{"}.$$

Assume the first case. The treatment of the second case is completely analogous in view of the symmetric definition of c ((I) and (II)).

Since $\chi' \leqq \chi \leqq \bar{\chi}$ we get that the j-interpolant of χ' and $\bar{\chi}$ forces $\check{\lambda} \in b$. This

j-interpolant is the direct extension of χ' determined by η and $\langle Q_1,\cdots,Q_l\rangle$ where $Q_1,\cdots,Q_l\in B$ (because $\chi\leqq\tilde{\chi}$). Hence by definition of F, $(\eta,\lambda,1)\in F(\{Q_1,\cdots,Q_l\})=E_l$, but χ clearly forces $\eta\in G\restriction j$, hence by definition of c (use definition (I)),

$$\chi\Vdash\lambda\in\text{"The realization of } c \text{ by } G\restriction j\text{"}$$

which contradicts our assumption. Thus we proved $\chi\Vdash b\in V[G\restriction j]$. □

Using Theorem 3.2 we can obtain information about the structure of the set of cardinals below κ as the set of cardinals $\leqq\kappa_j^{+(k-1)}$ is the same in $V[G]$ and in $V[G\restriction j]$. But $G\restriction j$ is $\mathcal{P}_j$ generic and $\kappa_1,\cdots,\kappa_j$ are inaccessible. Standard forcing arguments (see [23] for similar arguments or [6] p. 72, especially lemma 60) show that $V[G\restriction j]$ ω_1 is still a cardinal, the next cardinal is κ_1, then $\kappa_1^{+1},\cdots,\kappa_1^{+k}$, then $\kappa_2,\kappa_2^{+1},\cdots,\kappa_2^{+k},\kappa_3,\cdots,\kappa_j,\kappa_j^{+1},\cdots,\kappa_j^{+k}$. Hence the same pattern occurs in $V[G]$. Since κ has at least ω cardinals below it (each κ_i is a cardinal), below κ_j there are exactly $2+k(j-1)$ cardinals and the sequence $\langle\kappa_j\mid j<\omega\rangle$ is cofinal in κ, we get that in $V[G]$, κ is $\aleph_\omega$.

As for powers of cardinals below κ, again we can use Theorem 3.2 for $\mu<\kappa$ (hence $\mu\leqq\kappa_j$ for some $j<\omega$). The powerset of μ is the same in $V[G]$ and $V[G\restriction j]$. κ is inaccessible in V, hence also in $V[G\restriction j]$ because $G\restriction j$ is $\mathcal{P}_j$ generic and $\mathcal{P}_j$ has cardinality less than κ, κ is inaccessible in $V[G\restriction j]$. (See [9] for similar arguments.) Therefore 2^μ in $V[G\restriction j]$ is less than κ, but the same holds in $V[G]$. So we proved that κ is a strong limit cardinal in $V[G]$.

We have almost arrived at our destination. The only problem is that $\aleph_\omega$ of $V[G]$, namely κ, may satisfy $2^{\aleph_\omega}=\aleph_{\omega+1}$ because in V we had $2^\kappa=\kappa^{+k}$ and $\kappa^{+1},\cdots,\kappa^{+(k-1)}$ were collapsed. So we pass to a submodel of $V[G]$ which is $V[\langle\kappa_i\mid i<\omega\rangle,\langle F_i\mid i<\omega\rangle]$. The pair $\langle\kappa_i\mid i<\omega\rangle,\langle F_i\mid i<\omega\rangle$ can easily be coded as a set of ordinals. Denote this submodel by V_0. Since $G\restriction j$ for every $j<\omega$ is in V_0, V_0 and $V[G]$ have the same bounded subsets of κ, hence the same cardinals below κ. Therefore

$$V_0\vDash\kappa\text{ is }\aleph_\omega\wedge\kappa\text{ is a strong limit cardinal.}$$

Since $V\subseteq V_0$ and in V $\kappa^{+k}\leqq2^\kappa$ we get $V_0\vDash|\kappa^{+k}|\leqq2^{\aleph_\omega}$ (note that κ^{+k} is in the sense of V). Our final goal, therefore, will be to show that in V_0 κ^{+1}, $\kappa^{+2},\cdots,\kappa^{+k}$ are cardinals, hence we get

$$V_0\vDash2^{\aleph_\omega}\geqq\aleph_{\omega+k}.\wedge\aleph_\omega\text{ is a strong limit cardinal.}$$

Actually we can show that in V_0 $2^{\aleph_\omega}$ is exactly $\aleph_{\omega+k}$, but we shall not bother doing it. If we want to make sure that $2^{\aleph_\omega}=\aleph_{\omega+k}$ we can always force over V_0

with $\mathrm{Col}(\kappa^{+k},(2^{\aleph_\omega})^+)$ without changing the cardinals below κ^{+k} or the fact that $\aleph_\omega$ is a strong limit cardinal.

Let $0 \leqq i < k$. Our objective is to show that in V_0 $\kappa^{+(i+1)}$ is a cardinal. Define $V_i = V[\langle P_j \cap \kappa^{+i} \mid j < \omega \rangle, \langle F_j \mid j < \omega \rangle]$. Since obviously $V_0 \subseteq V_i$ the required statement follows from Theorem 3.3. Note that $V_{k-1} = V[G]$.

THEOREM 3.3. $V_i \models \kappa^{+(i+1)}$ *is a cardinal.*

PROOF. If $l \leqq i$ then the sequence $\langle P_j \cap \kappa^{+l} \mid j < \omega \rangle$ is in V_i, therefore by Lemma 3.1,

$$V_i \models \kappa^{+1}, \kappa^{+2}, \cdots, \kappa^{+i} \text{ have cofinality } \omega.$$

So none of $\kappa^{+1}, \kappa^{+2}, \cdots, \kappa^{+i}$ is a cardinal in V_i. Assume that $\kappa^{+(i+1)}$ is not a cardinal in V_i, hence it must be singular there. Let μ be the cofinality of $\kappa^{+(i+1)}$ in V_i. Since μ is a regular cardinal in V_i, $\mu < \kappa^{+(i+1)}$ we must have $\mu < \kappa$. (There are no cardinals between κ and $\kappa^{+(i+1)}$ so $\mu \leqq \kappa$, but we cannot have $\mu = \kappa$ because κ is singular.)

Let b be a name for a function from μ into $\kappa^{+(i+1)}$ whose range is cofinal in $\kappa^{+(i+1)}$, such that this function lies in $V_1 = V[\langle P_j \cap \kappa^{+i} \mid j < \omega \rangle, \langle F_j \mid j < \omega \rangle]$. By the remarks made in §1 we can assume that the name b is invariant under any automorphism of $\mathcal{P}$ which preserves some standard name for the pair $\{\langle P_j \cap \kappa^{+i} \mid j < \omega \rangle, \langle F_j \mid j < \omega \rangle\}$. We shall be interested in a particular group of automorphisms of $\mathcal{P}$, $\mathcal{G}$, for which it is obvious that any reasonable way of picking a standard name for $\{\langle P_j \cap \kappa^{+i} \mid j < \omega \rangle, \langle F_j \mid j < \omega \rangle\}$ will yield a name invariant under any automorphism from G.

Let $\mathcal{G}$ be the group of permutations of $\kappa^{+(k-1)}$ which are the indentity on κ^{+i}. $\mathcal{G}$ can be considered to be a group of automorphisms of $\mathcal{P}$, where we define the operation of $\Gamma \in \mathcal{G}$ on $\mathcal{P}$ as follows: If $P \in P_\kappa(\kappa^{+(k-1)})$ ΓP is $\Gamma'' P = \{\Gamma\alpha \mid \alpha \in P\}$. If $A \subseteq P_\kappa(\kappa^{+(k-1)})$ then ΓA is $\{\Gamma P \mid P \in A\}$. If $\pi \in \mathcal{P}$,

$$\pi = \langle P_1, \cdots, P_n, f_0, \cdots, f_n, A, H \rangle$$

then $\Gamma\pi$ is

$$\langle \Gamma P_1, \cdots, \Gamma P_n, f_0, \cdots, f_n, \Gamma A, H \cdot \Gamma^{-1} \rangle.$$

$\Gamma\pi$ is easily seen to be in $\bar{\mathcal{P}}$. (Note that Γ is the identity on κ, hence P and ΓP satisfy $P \cap \kappa = (\Gamma P) \cap \kappa$ therefore f_i, $i = 0, \cdots, n$ and $G \cdot \Gamma^{-1}(P)$ are still members of the right $\mathrm{Col}(\alpha, \beta)$ respectively.) The only clause which is not completely trivial is showing that if $A \in U$ then $\Gamma A \in U$. This follows from:

LEMMA 3.4. *Let Γ be a permutation of α, U is a normal ultrafilter on $P_\kappa(\alpha)$, then* $\{P \mid \Gamma P = P\} \in U$.

PROOF. Assume otherwise. Let $B = \{P \mid \Gamma P \neq P\} \in U$. For $P \in B$ there is $\beta_P \in P$ such that either $\Gamma\beta_P \notin P$ or $\Gamma^{-1}\beta_P \notin P$. By normality of U there exists $C \subseteq B$, $C \in U$, and $\beta \in \alpha$ such that for $P \in C$, $\beta_P = \beta$. By normality again pick $P \in C$ such that $\{\beta, \Gamma\beta, \Gamma^{-1}\beta\} \subseteq P$ which contradicts $\beta_P = \beta$. □

Since $\{P \mid P \in A,\ \Gamma P = P\} \subseteq \Gamma A$ we get that $\Gamma A \in U$. Γ defined on $\mathcal{P}$ is readily seen to be an automorphism, i.e., $\pi' \leqq \pi$ if and only if $\Gamma\pi' \leqq \Gamma\pi$ as one can verify by checking Definition 2.2 item by item. The fact that Γ is one to one and onto is equally routine. It is clear that any reasonable name for $\{\langle P_j \cap \kappa^{+i} \mid j < \omega\rangle, \langle F_j \mid j < \omega\rangle\}$ is invariant under $\Gamma \in \mathcal{G}$ because Γ is the identity on κ^{+i}, hence $P \cap \kappa^{+i} = (\Gamma P) \cap \kappa^{+i}$ and Γ when applied to a condition $\pi \in \mathcal{P}$ does not change its f part. Therefore we shall assume $\Gamma b = b$ for every $\Gamma \in \mathcal{G}$.

We need one extra lemma to finish the proof of Theorem 3.3.

LEMMA 3.5. *Let π, $\pi' \in \mathcal{P}$,*

$$\pi = \langle P_1, \cdots, P_n, f_0, \cdots, f_n, A, G\rangle, \qquad \pi' = \langle Q_1, \cdots, Q_n, f_0, \cdots, f_n, B, H\rangle$$

where

(a) *for* $1 \leqq j \leqq n$, $Q_j \cap \kappa^{+i} = P_j \cap \kappa^{+i}$,

(b) *for* $P \in A \cap B$, $H(P) = G(P)$,

then there exists $\Gamma \in \mathcal{G}$ such that $\Gamma\pi$ is compatible with π'.

PROOF. We can assume $i < k - 1$ otherwise π and π' are compatible because (a) implies $Q_j = P_j$ for $1 \leqq j \leqq n$. We define Γ piecewise on $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n$. Note that for $1 \leqq j \leqq n$ $|P_j| = (P_j \cap \kappa)^{+(k-1)} = (Q_j \cap \kappa)^{+(k-1)} = |Q_j|$ and that $|P_j \cap \kappa^{+i}| = (P_j \cap \kappa)^{+i} = (Q_j \cap \kappa)^{+i} = |Q_j \cap \kappa^{+i}| < |P_j| = |Q_j|$. Also $|P_1| < |P_2| < \cdots < |P_n|$, hence for $1 \leqq j < n$ we have

$$|P_{j+1} - P_j - \kappa^{+i}| = |(P_{j+1} - P_j) - (P_{j+1} \cap \kappa^{+i})| = (\kappa \cap P_j)^{+(k-1)}$$

(since $|P_j| < |P_{j+1}|$ and $|P_{j+1} \cap \kappa^{+i}| < |P_{j+1}|$). Similarly

$$|Q_{j+1} - Q_j - \kappa^{+i}| = (\kappa \cap Q_j)^{+(k-1)} = |P_{j+1} - P_j - \kappa^{+i}|.$$

Define Γ to be the indentity on κ^{+i}. On $P_1 - \kappa^{+i}$ pick any one to one function from $P_1 - P_1 \cap \kappa^{+i}$ onto $Q_1 - Q_1 \cap \kappa^{+i}$ (which exists by the equality of the cardinals), and any one to one function from $P_{j+1} - P_j - \kappa^{+i}$ onto $Q_{j+1} - Q_j - \kappa^{+i}$ for $1 \leqq j < n$. (Again we use $|P_{j+1} - P_j - \kappa^{+i}| = |Q_{j+1} - Q_j - \kappa^{+i}|$.) Extend Γ to all of $\kappa^{+(k-1)}$, by picking any one to one function from $\kappa^{+(k-1)} - P_n - \kappa^{+(k-1)}$ onto $\kappa^{+(k-1)} - Q_n - \kappa^{+(k-1)}$. Clearly $\Gamma P_j = Q_j$ (we use $P_j \cap \kappa^{+i} = Q_j \cap \kappa^{+i}$).

$\Gamma\pi$ is $\langle Q_1, \cdots, Q_n, f_1, \cdots, f_n, \Gamma A, G \cdot \Gamma^{-1}\rangle$. Let $C = \{P \mid P \in A \cap B, \Gamma P = P\}$. By Lemma 3.4 $C \in U$. Note that for $P \in C$ $H(P) = G(P) = G \cdot \Gamma^{-1}(P)$. $\langle Q_1, \cdots, Q_n, f_1, \cdots, f_n, C, H \restriction C\rangle$ is a common extension of $\Gamma\pi$ and of π', hence $\Gamma\pi$ and π' are compatible. □

We return to the proof of Theorem 3.3. Assume b is forced by the condition π to be realized as a function from μ to $\kappa^{+(i+1)}$ cofinal in $\kappa^{+(i+1)}$. Let n be the length of π. Without loss of generality we can assume that $\mu \leqq \kappa_j$ for some $j \leqq n$. We can apply Corollary 2.9 and get π', $\pi \leqq \pi'$, $\pi' = \langle P_1, \cdots, P_n, f_0, \cdots, f_n, A, G\rangle$, such that if $\pi' \leqq \pi''$, $\lambda \in \mu$ and $\pi'' \Vdash b\ (\check{\lambda}) = \check{\alpha}$ for some α, then the j-interpolant of π' and π'' forces the same statement. Fix $\lambda < \mu$, let

$$A_\lambda = \{\alpha \mid \text{For some } \pi' \leqq \pi'', \pi'' \Vdash \tau(\check{\lambda}) = \check{\alpha}\}.$$

Claim. $|A_\lambda| \leqq \kappa^{+i}$.

We postpone the proof of the claim, so assume the claim. $\bigcup_{\lambda<\mu} A_\lambda$ must be bounded in $\kappa^{+(i+1)}$ (note that the sequence $\{A_\lambda \mid \lambda < \mu\}$ is in V). Let δ be this bound, then clearly $\pi' \Vdash$ The range of b is bounded by δ, which contradicts the fact that $\pi \leqq \pi'$ forces b to be cofinal in $\kappa^{+(i+1)}$. Hence $\kappa^{+(i+1)}$ is regular in V_i.

Proof of the Claim. For each α in A_λ, pick $\pi' \leqq \pi_\alpha$ which forces $\tau(\check{\lambda}) = \check{\alpha}$. By the condition we imposed on π' we know that the j-interpolant of π' and π_α forces "$\tau(\check{\lambda}) = \check{\alpha}$". Hence without loss of generality we can assume that π_α is a j-direct extension of π'. Let $\eta_\alpha, Q_1^\alpha, \cdots, Q_{l_\alpha}^\alpha$ determine π_α as a j-direct extension of π' ($\eta_\alpha \in \mathcal{P}_j$). Assume $|A_\lambda| = \kappa^{+(i+1)}$. Since $|\mathcal{P}_j| < \kappa$, we have a subset of A, B, such that $|B| = \kappa^{+(i+1)}$ and for some η, l $\eta_\alpha = \eta$, $l_\alpha = l$ for every $\alpha \in B$. Similarly since κ is inaccessible $(\kappa^{+i})^{\underset{\sim}{\kappa}} = \kappa^{+i}$, the cardinality of possible sequences of the form $\langle Q_1^\alpha \cap \kappa^{+i}, Q_2^\alpha \cap \kappa^{+i}, \cdots, Q_l^\alpha \cap \kappa^{+i}\rangle$ is at most κ^{+i}. Therefore we can assume that for all $\alpha, \beta \in B$, $1 \leqq j \leqq l$

$$Q_j^\alpha \cap \kappa^{+i} = Q_j^\beta \cap \kappa^{+i}.$$

π_α, being a j-direct extension of π', has the form (at least for $\alpha \in B$)

$$\langle P_1, \cdots, P_n, Q_1^\alpha, \cdots, Q_l^\alpha, g_0, \cdots, g_{j-1}, f_j, \cdots, f_n, G(Q_1^\alpha), \cdots, G(Q_l^\alpha), B_\alpha, G \restriction B_\alpha\rangle$$

where $\langle g_0, \cdots, g_{j-1}\rangle = \eta$. Each $G(Q_j^\alpha)$ is a member of $\mathrm{Col}(\rho, \kappa)$ for some $\rho < \kappa$, which has cardinality κ. Therefore the cardinality of the different sequences of the form $G(Q_1^\alpha), \cdots, G(Q_l^\alpha)$ is at most $\kappa^l = \kappa$. Thus we can assume without loss of generality that for $\alpha, \beta \in B$ and for some fixed $h_1, \cdots, h_l$, $G(Q_j^\alpha) = G(Q_j^\beta) = h_j$. Fix $\alpha, \beta \in B$, $\alpha \neq \beta$, then

$$\pi_\alpha = \langle P_1, \cdots, P_n, Q_1^\alpha, \cdots, Q_l^\alpha, g_0, \cdots, g_{j-1}, f_j, \cdots, f_n, h_1, \cdots, h_l, B_\alpha, G \restriction B_\alpha \rangle,$$

$$\pi_\beta = \langle P_1, \cdots, P_n, Q_1^\beta, \cdots, Q_l^\beta, g_0, \cdots, g_{j-1}, f_j, \cdots, f_n, h_1, \cdots, h_l, B_\beta, G \restriction B_\beta \rangle.$$

We see that the conditions of Lemma 3.5 are satisfied if we substitute π_α for π and π_β for π'. (Note that $Q_j^\alpha \cap \kappa^{+i} = Q_j^\beta \cap \kappa^{+i}$ for $1 \leqq j \leqq l$.) Therefore we can find $\Gamma \in \mathscr{G}$ such that $\Gamma\pi_\alpha$ is compatible with π_β.

By definition of π_α and π_β we have

$$\pi_\alpha \Vdash b(\check{\lambda}) = \check{\alpha}, \qquad \pi_\beta \Vdash b(\check{\lambda}) = \check{\beta}.$$

Since b is invariant under Γ we get from the first forcing relation $\Gamma\pi_\alpha \Vdash b(\check{\lambda}) = \check{\alpha}$, but this contradicts the fact that π_β and $\Gamma\pi_\alpha$ are compatible. □

This completes the proof of Theorem 1 (a).

§4. Making $2^{\aleph_\omega}$ a little bit larger

In this section we shall indicate how to modify the construction so as to get a model in which $\aleph_\omega$ is a strong limit cardinal while $2^{\aleph_\omega} = \aleph_{\omega+\omega+1}$. We do not know how to make $2^{\aleph_\omega}$ even larger, but we expect it to be possible. Maybe the methods of [13] can be used. The exact assumption that we need about the ground model V is that it contains a cardinal κ such that

(a) $2^\kappa = \kappa^{+(\omega+1)}$,

(b) κ is κ^{+n} supercompact for every $n < \omega$.

Again [21] or [14] can be used to get such a model from κ which is $\kappa^{+(\omega+1)}$ supercompact. Following the lead of §§2 and 3 we change the cofinality of κ as well as κ^{+n} for every $n < \omega$ to ω, hence κ^{+n} has cardinality κ in $V[G]$. Simultaneously κ is made $\aleph_\omega$ of $V[G]$. In the appropriate submodel we are able to show that κ^{+n} (hence $\kappa^{+\omega}$ as their limit) together with $\kappa^{+\omega+1}$ are still cardinals, thus this particular submodel is a witness for the truth of Theorem 1 (b).

Since κ is assumed to be κ^{+n} supercompact for every $n < \omega$, fix for every $n < \omega$ U_n, a normal ultrafilter over $P_\kappa(\kappa^{+n})$. Note that if $P \in P_\kappa(\kappa^{+n})$ then $P \in P_\kappa(\kappa^{+l})$ for every $n < l$. Analogously to §2 we define: $D_n = \{P \mid P \cap \kappa \text{ is an inaccessible cardinal, } \operatorname{otp}(P) = (P \cap \kappa)^{+n}\}$.

Definition 4.1. The set of forcing conditions $\mathscr{P}$ is the set of sequences of the form $\pi = \langle P_1, \cdots, P_n, f_0, \cdots, f_n, \langle A_i \mid n < i < \omega \rangle, \langle G_i \mid n < i < \omega \rangle \rangle$ where

(a) for $1 \leqq i \leqq n$ $\quad P_i \in P_\kappa(\kappa^{+i})$, $P_i \in D_i$,
for $1 \leqq i \leqq n$ $\quad P_i \subsetneq P_{i+1}$;

(b) $f_0 \in \operatorname{Col}(\omega_1, \kappa \cap P_1)$,

for $1 \leqq i < n$ $\quad f_i \in \mathrm{Col}((\kappa \cap P_i)^{+(i+1)}, \kappa \cap P_{i+1})$,
$f_n \in \mathrm{Col}((\kappa \cap P_n)^{+(n+1)}, \kappa)$;

(c) $A_j \in U_j$ for $n < \omega$, and for all $P \in A_j$ $\quad P_n \subsetneq P$,
$f_n \in \mathrm{Col}((\kappa \cap P_n)^{+(n+1)}, \kappa \cap P)$;

(d) G_j is a function defined on A_j where $G_j(P) \in \mathrm{Col}((\kappa \cap P)^{+(j+1)}, \kappa)$, where for every $Q \in A_{j+1}$ such that $P \subsetneq Q$, $G_j(P) \in \mathrm{Col}((\kappa \cap P)^{+(j+1)}, \kappa \cap Q)$.

The intuitive motivation is that $\langle \mathrm{Sup}(P_i \cap \kappa^{+i}) \mid i < n \rangle$ is an approximation to a sequence that in the generic filter will be cofinal in κ^{+j}. A_j is the set of possible candidates for being the j-member of the sequence. f_i is partial information about collapsing most of the cardinals between $\kappa \cap P_i$ and $\kappa \cap P_{i+1}$ leaving just finitely many of them. $G_j(P)$ is a commitment we make about f_j if we decide that the j-th member of the sequence will be P.

We shall not bother the reader with all the details which are anyhow just a minor change of the development in §§2 and 3. We shall just indicate the major steps that should be taken in order to finish the proof:

(A) The intuitive motivation should suffice to define the partial order relation on $\mathcal{P}$, modeled after Definition 2.2. Similarly the definitions of j-direct extensions, j-length preserving extension and j-interpolant naturally generalize to the present case.

(B) Theorem 2.6 and Corollary 2.9 hold as stated with the obvious changes in the proofs.

(C) If we use $\mathcal{P}$ as the set of forcing conditions, in the model $V[G]$ that we get, G generates a sequence $\langle P_i \mid i < \omega \rangle$ such that $P_i \in P_\kappa(\kappa^{+i})$ and for $n < \omega$ $\langle \mathrm{Sup}(P_i \cap \kappa^{+n}) \mid i < \omega \rangle$ is cofinal in κ^{+n} and a sequence of collapsing maps $\langle F_j \mid j < \omega \rangle$ such that in $V[\langle P_i \cap \kappa \mid i < \omega \rangle, \langle F_i \mid i < \omega \rangle]$ κ is $\aleph_\omega$. Hence each κ^{+n} is collapsed to κ and therefore their limit ($\kappa^{+\omega}$) is collapsed. Also in $V[G]$, κ (i.e. $\aleph_\omega$) is a strong limit cardinal. This follows from the appropriate version of Theorem 3.2. In the proof we need a substitute for Theorem 1.2 which is supplied by the following theorem. We shall state it in a more general version than we need for this section, but it will be used in the next section. Some notations before the statement of the theorem: Let $\alpha < \kappa$, $A_i \subseteq P_\kappa(\lambda_i)$ for $1 \leqq i \leqq n$, $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$. $A_1 \otimes A_2 \cdots \otimes A_n$ is the set

$$\{\langle P_1, \cdots, P_n \rangle \mid P_i \in A_i, 1 \leqq i \leqq n, P_i \subsetneq P_{i+1} \text{ for } 1 \leqq i \leqq n-1\}.$$

Given: $\langle A_\beta \mid \beta < \alpha \rangle$ where $A_\beta \subseteq P_\kappa(\lambda_\beta)$ where $\lambda_\beta \leqq \lambda_\gamma$ if $\beta \leqq \gamma$ then $\langle A_\beta \mid \beta < \alpha \rangle^{<\omega}$ is $\bigcup \{A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_n} \mid \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha\}$.

THEOREM 4.2. *Let $\langle U_\beta \mid \beta < \alpha \rangle$ be a sequence of ultrafilters where U_β is a normal ultrafilter on $P_\kappa(\lambda_\beta)$, for some λ_β, where $\lambda_\gamma \leqq \lambda_\beta$ if $\gamma \leqq \beta$. Let F be a portion of $\langle P_\kappa(\lambda_\beta) \mid \beta < \alpha \rangle^{<\omega}$ into less than κ parts. Then there exists a sequence of sets $\langle A_\beta \mid \beta < \alpha \rangle$ such that $A_\beta \in U_\beta$ and for $\alpha_1 < \cdots < \alpha_n < \alpha$, F is constant on $A_{\alpha_1} \otimes A_{\alpha_2} \cdots \otimes A_{\alpha_n}$.*

The proof is very similar to analogous results for measurable cardinals (see Rowbottom [18]). We use the closure of normal ultrafilters under diagonal intersections (see §1). We shall omit the details. We apply the theorem for the case $\alpha = \omega$, $\lambda_n = \kappa^{+n}$.

(D) The submodel of $V[G]$ we are interested in is V_0 where

$$V_0 = V[\langle P_i \cap \kappa \mid i < \omega \rangle, \langle F_i \mid i < \omega \rangle].$$

Clearly κ is $\aleph_\omega$ in V_0 and it is a strong limit cardinal. The crucial point is that κ^{+n} $(n < \omega)$, $\kappa^{+\omega}$ and $\kappa^{+(\omega+1)}$ are cardinals in V_0. Otherwise let $\alpha \leqq \omega + 1$ be the first cardinal such that $\kappa^{+\alpha}$ is not a cardinal in V_0. α must be a successor cardinal (otherwise $\kappa^{+\alpha}$ is a cardinal as the limit of $\langle \kappa^{+\beta} \mid \beta < \alpha \rangle$. $\alpha = \beta + 1$. Let $V_\beta = V[\langle P_i \cap \kappa^{+\beta} \mid i < \omega \rangle, \langle F_i \mid i < \omega \rangle]$. Clearly $V_0 \subseteq V_\beta$ and an argument which essentially repeats the argument in §3 (using the group of permutations of $\kappa^{+\omega}$ which are constant on $\kappa^{+\beta}$) shows that $V_\beta \models \kappa^{+(\beta+1)}$ is a cardinal, hence the same holds in V_0, which contradicts our assumption. □

This completes the proof of Theorem 1 (b).

§5. How to handle cardinals other than $\aleph_\omega$

Our proof so far was in many aspects modeled after Prikry's forcing, for changing the cofinality of a measurable cardinal to ω. For handling singular cardinals other than $\aleph_\omega$ and in particular cardinals of cofinality greater than ω, we need a generalization of Prikry's forcing by which we can change the cofinality of a regular cardinal to some value different from ω, without collapsing any cardinals. This is exactly what was done in [12]. The basic idea of this section, which sketches the proof of Theorem 2, is to combine the methods of [12] with the methods of §§2 and 3. The following "indentity" can probably describe the situation:

$$\frac{\text{§§2, 3}}{\text{Prikry Forcing}} = \frac{\text{§5}}{[12]}$$

As in §4 we shall mainly describe the forcing notion which generates a model $V[G]$ which is a witness for Theorem 2, and we shall let the reader bother about the details if he has no better recreation. So let κ, β, γ be as in Theorem 2 (in the introduction) and $\beta = \mu + \Sigma_{\alpha<\rho}(\delta_\alpha)$, where $\langle \delta_\alpha \mid \alpha < \rho \rangle$ is an additive partition of β with $\sup\langle \delta_\alpha \mid \alpha < \rho\rangle + 1 \geqq \gamma$. Since γ is a successor ordinal, $\gamma = \delta + 1$ where $\delta \leqq \sup\langle \delta_\alpha \mid \alpha < \rho \rangle$. Without loss of generality we may assume $|\beta| \leqq \aleph_\mu$ if $|\beta| < \aleph_{|\beta|}$ because under the assumption $|\beta| < \aleph_{|\beta|}$, $|\beta| = \aleph_\chi$ for some $\chi < \beta$ and we can assume $\chi \leqq \mu$.

Definition 5.1. Let $U, \tilde{U}$ be normal ultrafilters on $P_\kappa(\rho)$, $P_\kappa(\eta)$ respectively, where $\rho \leqq \eta$. We say that U is majored by $\tilde{U}$ ($U < \tilde{U}$) if when we form the ultrapower $V^{P_\kappa(\eta)}/\tilde{U}$ and collapse it to a transitive class M (see [19]) then $U \in M$.

As in [12] (after which Definition 5.1 is modeled) we can show that $<$ is a partial order on normal ultrafilters. (In fact it is even well founded.) The following lemma is due to Solovay [17].

Lemma 5.2. *Let κ be a supercompact cardinal $\kappa \leqq \lambda$ and $A \subseteq P(P_\kappa(\lambda))$ then there exists a normal ultrafilter U on $P_\kappa(\lambda)$ such that $A \in V^{P_\kappa(\lambda)}/U$. (We do not distinguish between $V^{P_\kappa(\lambda)}/U$ and its transitive isomorph.)*

Using Lemma 5.2 and given a nondecreasing sequence of ordinals $\langle \lambda_\alpha \mid \alpha < \rho \rangle$ κ which is a supercompact cardinal and $\rho \leqq \lambda_0$, we can find an increasing sequence of normal ultrafilters $U_0 < U_1 < U_2 \cdots < U_\alpha < \cdots$ $(\alpha < \rho)$ where U_α is a normal ultrafilter on $P_\kappa(\lambda_\alpha)$. We simply construct the sequence by induction using Lemma 5.2. We note that given $\leqq \lambda$ ultrafilters, each of which is an ultrafilter on $P_\kappa(\mu)$ for some $\mu \leqq \lambda$, we can easily code the sequence as a subset of $P(P_\kappa(\lambda))$. This last argument together with [21] shows the existence of a Cohen extension of M, V (which from now on will be our ground model) in which

(a) $2 = \kappa^{+\gamma}$,

(b) there exists an $<$ increasing sequence $\langle U_\alpha \mid \alpha < \rho \rangle$ where U_α is a normal ultrafilter on $P_\kappa(\kappa^{+\delta_\alpha})$, where the sequence $\langle \delta_\alpha \mid \alpha < \rho \rangle$ is the additive partition of β fixed above.

We need a few more observations and notations before we can describe the forcing conditions. Let U be a normal ultrafilter on $P_\kappa(\lambda)$ for some ordinal λ. The transitive isomorph of $V^{P_\kappa(\lambda)}/U$ is M_U. For every ordinal $\alpha \leqq \delta$ define $\alpha\colon P_\kappa(\lambda) \to \kappa$ by $\alpha(P) = \mathrm{otp}(P \cap \alpha)$. (No confusion can possibly arise from denoting the function and the ordinal by the same name.) It is well known that the function α represents in the ultrapower $V^{P_\kappa(\lambda)}/U$ exactly the ordinal α (see [17] or [11]). Therefore if $\tilde{U} < U$ (which means that $\tilde{U} \in M_U$), where $\tilde{U}$ is a normal ultrafilter on $P_\kappa(\alpha)$ for $\alpha \leqq \lambda$, there is a function in $V^{P_\kappa(\lambda)}$ which represents $\tilde{U}$. Let $\tilde{F}(U, \tilde{U})$ be this function. Clearly the domain of $\tilde{F}$ is $P_\kappa(\lambda)$ and since $M_U \models \tilde{U}$ is a normal ultrafilter on $P_\kappa(\alpha)$, then by Łos Theorem ([10])

$$D(U, \tilde{U}) = \{S \mid S \in P_\kappa(\lambda), \tilde{F}(U, \tilde{U})(S) \text{ is a normal ultrafilter on } P_{\kappa(S)}(\alpha(S))\} \in U.$$

Since $\alpha(S)$ is the order type of $S \cap \alpha$, then if $\tilde{F}(U, \tilde{U})(S)$ is a normal ultrafilter on $P_{\kappa(S)}(\alpha(S))$, it naturally induces an equivalent normal ultrafilter on $P_{\kappa(S)}(S \cap \alpha)$. Note that $P_{\kappa(S)}(S \cap \alpha)$ is simply $\{Q \mid Q \in P_\kappa(\alpha),\ Q \subsetneq S\}$. Denote this equivalent normal ultrafilter by $F(U, \tilde{U})(S)$.

Let $\tilde{\tilde{U}}$ be a third ultrafilter which is a normal ultrafilter on $P_\kappa(\alpha')$ where $\alpha' \leqq \alpha$ and $\tilde{\tilde{U}} < \tilde{U}$. It is easy to check that $\tilde{\tilde{U}} \in M_U$ (it amounts to the statement that $<$ is transitive), as well as $F(\tilde{U}, \tilde{\tilde{U}}) \in M_U$. Also it can be verified that

$$M_U \models \tilde{\tilde{U}} < \tilde{U},\ \tilde{\tilde{U}} \text{ is represented in } M^{P_\kappa(\lambda)}/\tilde{U} \text{ by } \tilde{F}(\tilde{U}, \tilde{\tilde{U}}).$$

Remember that $\tilde{F}(U, \tilde{\tilde{U}})(S)$ and $\tilde{F}(U, \tilde{U})(S)$ are naturally isomorphic to $F(U, \tilde{\tilde{U}})(S)$ and $\tilde{F}(U, \tilde{U})(S)$ respectively. Moreover, $F(\tilde{U}, \tilde{\tilde{U}})$ as a member in M_U is represented (at least up to a set in $\tilde{U}$) in $V^{P_\kappa(\lambda)}/U$ by the function $f(S) = F(\tilde{U}, \tilde{\tilde{U}}) \restriction P_{\kappa(S)}(S \cap \alpha)$.

In view of Łos Theorem we get that $D(U, \tilde{U}, \tilde{\tilde{U}}) \in U$, where $D(U, \tilde{U}, \tilde{U})$ is the set of all $S \in P_\kappa(\lambda)$ such that

(a) $F(U, \tilde{\tilde{U}})(S) < F(U, \tilde{U})(S)$, $S \in D(U, \tilde{U}) \cap D(U, \tilde{\tilde{U}})$,

(b) $F(U, \tilde{\tilde{U}})(S)$ is represented in $V^{P_\kappa(S \cap \alpha)}/F(U, \tilde{U})(S)$ by the function $F(\tilde{U}, \tilde{\tilde{U}}) \restriction P_{\kappa(S)}(S \cap \alpha)$. Note also that if $A \subsetneq P_\kappa(\alpha)$ then $A \in \tilde{U}$ if and only if $\{S \mid A \cap P_{\kappa(S)}(S \cap \alpha) \in F(U, \tilde{U})(S)\} \in U$.

Let us go back to our sequence $U_0 < U_1 < \cdots < U_\alpha < \cdots \quad (\alpha < \rho)$. For $\alpha < \rho$ define

$$D_\alpha = \bigcap_{\eta < \alpha} D(U_\alpha, U_\eta) \cap \bigcap_{l < \eta < \alpha} D(U_\alpha, U_\eta, U_l) \cap$$

$\{P_\kappa \in P_\kappa(\kappa^{+\delta_\alpha}) \mid \kappa \cap P$ is an inaccessible cardinal, which is $\kappa(P)$, $\operatorname{otp}(P) = (P \cap \kappa)^{+\delta_\alpha}\}$,

$$D_\alpha \in U. \text{ (We use the remarks above and Lemma 1.1.)}$$

Definition 5.2. The set of forcing conditions P is the set of all tuples of the form

$$\pi = \langle \Gamma, P_{\alpha_1}, \cdots, P_{\alpha_n}, f_0, f_{\alpha_1}, \cdots, f_{\alpha_n}, \langle A_\alpha \mid \alpha \in \rho - \Gamma \rangle, \langle G_\alpha \mid \alpha \in \rho - \Gamma \rangle \rangle$$

where

(a) Γ is a finite subset of ρ, $0 \in \Gamma$ and $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_n$ is a monotone enumeration of the elements of Γ.

(b) $P_{\alpha_i} \in D_{\alpha_i}$ for $1 \leqq i \leqq n$, $P_{\alpha_1} \subsetneq P_{\alpha_2} \subsetneq \cdots \subsetneq P_{\alpha_n}$.

(c) $f_0 \in \mathrm{Col}(\aleph_{\mu+1}, \kappa(P_{\alpha_1}))$, $f_{\alpha_i} \in \mathrm{Col}(\mathrm{otp}(P))^+, \kappa(P_{\alpha_{i+1}}))$ for $1 \leqq i \leqq n$ and $f_{\alpha_n} \in \mathrm{Col}(\mathrm{otp}(P_{\alpha_n})^+, \kappa)$.

(d) If $\alpha_n < \lambda \in \rho - \Gamma$, $A_\lambda \in U_\lambda$, $A_\lambda \subseteq D_\lambda$. For every Q in A_λ $P_{\alpha_n} \subsetneq Q$ and $f_n \in \mathrm{Col}(\mathrm{otp}(P_{\alpha_n})^+, \kappa(Q))$.

(e) If $\lambda \in \rho - \Gamma$, $\alpha < \lambda < \alpha_{i+1}$ for some $0 \leqq i < n$ then $A_\lambda \in F(U_{\alpha_{i+1}}, U_\lambda)(P_{\alpha_{i+1}})$, $A_\lambda \subseteq D_\lambda$. For every Q in A_γ $P_{\alpha_i} \subsetneq Q$ and $f_{\alpha_i} \in \mathrm{Col}(\mathrm{otp}(P_{\alpha_i})^+, \kappa(Q))$.

(f) G_λ is a function defined on A_λ such that:

(1) If $\alpha_n < \lambda \in \rho - \Gamma$, $G_\lambda(P) \in \mathrm{Col}(\mathrm{otp}(P)^+, \kappa)$ and if $\lambda < \eta \in \rho - \Gamma$, $P \subsetneq Q \in A_\eta$ then $G_\lambda(P) \in \mathrm{Col}(\mathrm{otp}(P)^+, \kappa(Q))$.

(2) If $\alpha_i < \lambda < \alpha_{i+1}$ for some $0 \leqq i < n$ then $G_\lambda(P) \in \mathrm{Col}(\mathrm{otp}(P)^+, \kappa(P_{\alpha_{i+1}}))$ and if $\lambda < \eta < \alpha_{i+1}$, $P \subsetneq Q \in A_\eta$ then $G_\lambda(P) \in \mathrm{Col}(\mathrm{otp}(P)^+, \kappa(Q))$.

The intuitive motivation to Definition 5.2 is very similar to the one given to Definition 2.1 or 4.1. See also [12]. We want our generic filter to generate ρ sequences cofinal in each of the cardinals $\kappa, \kappa^{+1}, \kappa^{+2}, \cdots, \kappa^{+\delta}$ respectively. $\langle P_{\alpha_i} \mid \alpha_i \in \Gamma \rangle$ is a finite approximation to a function from α to $P_\kappa(\kappa^{+\delta})$ such that when the whole function $\langle P_\alpha \mid \alpha < \rho \rangle$ is realized then $\langle \mathrm{Sup}(P \cap \kappa^{+\eta}) \mid \alpha < \rho \rangle$ is a cofinal sequence in $\kappa^{+\eta}$ for every successor η, such that $\eta \leqq \delta$. A_λ is the set of possible candidates for being the λ-th member of the sequence $\langle P_\alpha \mid \alpha < \rho \rangle$. Since the sequence should be increasing with respect to $\subsetneq$, once we decided what is P_{α_i} and $P_{\alpha_{i+1}}$, if $\alpha_i < \lambda < \alpha_{i+1}$, P_λ should satisfy $P_{\alpha_i} \subsetneq P_\lambda \subsetneq P_{\alpha_{i+1}}$. Hence A_γ is not required to be in U_γ but in some "projection" of it to the set $\{Q \mid Q \in P_\kappa(\kappa^{+\delta_\lambda}),\ Q \subsetneq P_{\alpha_{i+1}}\}$. This projection is given by $F(U_{\alpha_{i+1}}, U_\lambda)(P_{\alpha_{i+1}})$. The requirement $P_{\alpha_i} \in D_{\alpha_i}$ guarantees that these "projections" cohere, namely $F(U_{\alpha_{i+1}}, U_\lambda)(P_{\alpha_{i+1}})$ is also the "projection" of $F(U_\eta, U_\lambda)(P_\eta)$ to $P_{\alpha_{i+1}}$ if $\eta < \alpha_{i+1}$. (This also applies to future P_η's, , namely for $\eta \notin \Gamma$, hence the value of P_η will be determined by some future condition.) $f_0, f_{\alpha_1}, \cdots, f_{\alpha_n}$ gives partial information about collapsing maps that eventually will make κ to $\aleph_\beta$. $G_\lambda(P)$ is a partial information about f_λ which we must use if we decide to use P as the λ-th member of the sequence. We omit the definition of the order relation on $\mathcal{P}$.

As in [12] and in §3 we can show that a V generic filter over $\mathcal{P}$ generates a sequence $\langle P_\lambda \mid \lambda < \rho \rangle$ such that $\langle \mathrm{Sup}(P \cap \kappa^{+\eta}) \mid \alpha < \rho \rangle$ is cofinal in $\kappa^{+\eta}$ for every successor η, $\eta \leqq \delta$. Hence, since every cardinal below $\kappa^{+\gamma}$ and above κ is either successor or a singular cardinal we get that $\kappa^{+\eta}$ $(\eta \leqq \delta)$ are all collapsed to κ.

Since we want to find what happens below κ, we need the definitions of λ-direct extension, λ-length preserving extension and λ-interpolant for $\lambda < \rho$.

DEFINITION 5.3. Let π be as in Definition 5.2, $\lambda < \rho$. Then $\pi \restriction \lambda$ (the restriction of π to λ) is

$$\langle \Gamma \cap \lambda, P_{\alpha_1}, \cdots, P_{\alpha_{j-1}}, f_0, \cdots, f_{\alpha_{j-1}}, \langle A_\eta \mid \eta < \lambda,\ \eta \in \Gamma \rangle, \langle G_\eta \mid \eta < \lambda,\ \eta \in \Gamma \rangle \rangle,$$

where j is the minimal such that $\lambda \leqq \alpha_j$, if there exists such α_j otherwise $j = n + 1$.

(Note that if $\lambda < \alpha_j$ for some α_j then we have less than κ possible restrictions for λ of extensions of π.)

DEFINITION 5.4. Let

$$\pi' = \langle \Gamma', P_{\beta_1}, \cdots, P_{\beta_l}, f_0, \cdots, f_{\beta_l}, \langle \beta_\eta \mid \eta \in \rho - \Gamma' \rangle, \langle H_\eta \mid \eta \in \rho - \Gamma \rangle \rangle,$$

π, λ as before. $\pi \leqq \pi'$. then π' is a λ-direct extension of π if

(a) For all $\beta_l \in \Gamma' - (\Gamma \cup \delta)$, $f_{\beta_l} = G_{\beta_l}(P_{\beta_l})$.

(b) For $\eta \in \rho - \Gamma'$ and $\beta_l < \eta$, $\lambda \leqq \eta < \rho$ then $B_\eta = \{Q \mid Q \in A_\eta, P_{\beta_l} \subsetneq Q\}$.

(c) For $\eta \in \rho - \Gamma'$, $\lambda \leqq \eta$, $\beta_j < \eta < \beta_{j+1}$ for some $1 \leqq j \leqq l$, $B_\eta = \{Q \mid Q \in A_\eta,\ P_{\beta_j} \subsetneq Q \subsetneq P_{\beta_{j+1}}\}$.

(d) For $\eta \in \rho - \Gamma'$, $\lambda \leqq \eta$, $H_\eta = G_\eta \restriction B_\eta$.

DEFINITION 5.5. Let π, π', λ be as in Definition 5.4, $\pi \leqq \pi'$ π is a λ-length preserving extension of π if

(a) $\Gamma = \Gamma'$,

(b) $\pi \restriction \lambda = \pi' \restriction \lambda$.

The λ-interpolant of π and π' if $\pi \leqq \pi'$ is defined as in §2. Once we made these definitions, Lemma 2.5, Theorem 2.6 and Corollary 2.9 go through (replace j by λ) with the obvious modification. For instance, in Corollary 2.9 change $\mu \leqq \kappa_j^{+(k-1)}$ to $\mu \leqq \mathrm{otp}(P_\lambda)$. Some additional arguments are needed in the proofs. Since they do not amount to more than the merging of the arguments in [12] and the proofs in §2 we omit further details.

For $\lambda < \rho$ define $\mathcal{P}_\lambda$ as $\{\pi \restriction \lambda \mid \pi \in \mathcal{P}\}$ and for $G \subseteq \mathcal{P}$, $G \restriction \lambda$ is $\{\pi \restriction \lambda \mid \pi \in G\}$. If G was V generic over $\mathcal{P}$, $G \restriction \lambda$ is generic over $\mathcal{P}_\lambda$. Note that it is not true here that $G \restriction \lambda$ is isomorphic to a finite product of collapses as was the case in §3, but Theorem 3.2 still holds if we replace j by λ. Using this we are able to show that no cardinal below λ is collapsed unless we specifically introduce conditions for collapsing it via some $\mathrm{Col}(\alpha, \eta)$. Similarly we can show that no new subsets of β are introduced, and that κ is a strong limit cardinal. We have to check that κ is $\aleph_\beta$ in the extension.

Note that below $\kappa(P_1)$ we left $\mu + 1$ cardinals, and the set of cardinals η such that $\kappa(P_\lambda) \leqq \eta < \kappa(P_{\lambda+1})$ has order type $\delta_\lambda + 1$, hence κ is $\aleph_\eta$ where $\eta = \mu + 1 + \Sigma_{\lambda<\rho}(\delta_\lambda + 1)$. We have to show:

LEMMA 5.6. *Under the conditions of Theorem* 2, $\eta = \beta$.

PROOF. $\mu + \Sigma_{\lambda<\rho}\delta_\lambda$ is by assumption β. An easy exercise in ordinal arithmetic will show that since ρ is a limit ordinal, $\mu + \Sigma_{\lambda<\rho}\delta_\lambda = (\mu + 1) + \Sigma_{\lambda<\rho}(\delta_\lambda + 1)$. □

Again the model $V[G]$ is not our final model. Let $\langle F_\alpha \mid \alpha < \rho \rangle$ be the sequences of the collapsing maps generated by the generic filter. Let V_0 be $V[\langle \kappa(P_\alpha) \mid \alpha < \rho \rangle, \langle F_\alpha \mid \alpha < \rho \rangle]$. Clearly in V_0, κ is $\aleph_\beta$ and it is a strong limit cardinal. Also $2^\kappa > \kappa^{+\gamma}$ ($\kappa^{+\gamma}$ in the sense of V). We finish this sketch of the proof of Theorem 2 by showing that for $\eta \leqq \gamma$, $\kappa^{+\eta}$ is a cardinal in V_0. Assume otherwise. Let $\kappa^{+\lambda}$ be the first among $\{\kappa^{+\eta} \mid \eta \leqq \gamma\}$ which is not a cardinal in V_0. λ is of course a successor ordinal, since the limit of cardinals is a cardinal. Therefore in V_0 it should be singular. Let η be its cofinality in V_0. If $\eta < \kappa$ then arguments similar to those we had in §3 yield a contradiction. If $\kappa \leqq \eta$ then $\eta = \kappa^{+\phi}$ for some $\phi < \lambda$, and since η has to be regular in V_0 (and κ is not), ϕ is a successor ordinal, $\phi < \lambda$. Let V_1 be $V[\langle P_\alpha \cap \kappa^{+\phi} \mid \alpha < l \rangle, \langle F_\alpha \mid \alpha < \rho \rangle]$. In V_1 all the cardinals $\leqq \kappa^{+\phi}$ are collapsed, hence the cofinality of $\kappa^{+\lambda}$ in V_1 is less than κ. Once again the argument of §3 applies, where we use automorphisms of $\mathcal{P}$ generated by permutations of $\kappa^{+\delta}$ which are constant on $\kappa^{+\phi}$. In the argument we have to find the cardinality of $P_\kappa(\kappa^{+\phi})$. Solovay [24] gives the answer: Since $\kappa^{+\phi}$ is regular (ϕ is a successor ordinal) and κ is $\kappa^{+\phi}$ supercompact we get that $|P_\kappa(\kappa^{+l})| = \kappa^{+l}$. □

This completes the proof of Theorem 2.

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